Independence preserving property and integrable systems

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Based on joint works with D.Croydon and R. Uozumi

Main question

Does the independence preserving property (IP property) have anything to do with the integrability?

Motivation:

- The IP property is quite fundamental and classical, but still not well understood. Relations to the integrability may be useful.
- The IP property may give a new and universal perspective for the relation between various deterministic/stochastic integrable systems.
- The IP property may be useful to find a new integrable model (or a new Yang-Baxter map).

Kac-Bernstein theorem

Let $F(x,y) = (x+y, x-y) : \mathbb{R}^2 \to \mathbb{R}^2$.

Kac-Bernstein theorem (1939,1941)

Suppose X, Y are two independent non-constant random variables and $(U, V) := F(X, Y)$ are also independent.
Then, there exists $a, b \in \mathbb{R}$ and $\sigma > 0$ such that $X \sim \mathrm{N}(a, \sigma)$, $Y \sim \mathrm{N}(b, \sigma)$, where $\mathrm{N}(a, \sigma)$ is the normal distribution with mean a and variance σ.

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There are a number of applications of this result in physics, statistics, ...
There are many generalizations of the result in various directions.

1. Independence preserving property
2. Classical integrable systems
3. Stochastic integrable models
4. Yang-Baxter map

Independence preserving property

- $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$: measurable spaces
- $F : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$: measurable bijection
- F has an independence preserving property (IP property) if there exists a quadruplet of non-dirac probability measures $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ (on $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively) satisfying $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$.
- In other words, there exist non-constant independent random variables X, Y such that U, V are also independent where $(U, V) = F(X, Y)$

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- In other words, there exist non-constant independent random variables X, Y such that U, V are also independent where $(U, V) = F(X, Y)$

Basic question

For which F, the independence preserving property holds? For such F, can we characterize all solutions $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ for $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$?

* For $F(x, y) = (f(x), g(y))$ or $F(x, y) = (f(y), g(x))$, F trivially has the IP property. Also, any coordinate-wise change of variables do not change the property, as well as $F \to F^{-1}$ and $F \to F \circ \pi$ where $\pi(x, y) = (y, x)$. These induce a natural equivalence relation in the class of functions F.

Known results

- $F_{\mathrm{N}} : \mathbb{R}^2 \to \mathbb{R}^2 : F_{\mathrm{N}}(x, y) = (x + y, x - y)$: Normal distribution
  (Kac 1939, Bernstein 1941),
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{N}(a, c), \mathrm{N}(b, c), \mathrm{N}(a + b, 2c), \mathrm{N}(a - b, 2c))$
- $F_{\mathrm{Ga}} : \mathbb{R}_+^2 \to \mathbb{R}_+^2 : F_{\mathrm{Ga}}(x, y) = \left(x + y, \frac{x}{y}\right)$: Gamma distribution
  (Lukacs, 1955)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{Ga}(a, c), \mathrm{Ga}(b, c), \mathrm{Ga}(a + b, c), \mathrm{Be}'(a, b))$
- $F_{\mathrm{Exp}} : \mathbb{R}^2 \to \mathbb{R}^2 : F_{\mathrm{Exp}}(x, y) = (\min\{x, y\}, x - y)$: Exponential /Geometric distribution
  (Ferguson, 1965, Crawford, 1966)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{sExp}(a, c), \mathrm{sExp}(b, c), \mathrm{sExp}(a + b, c), \mathrm{AL}(a, b))$, or
  $(\mathrm{ssGeo}(p, M, m), \mathrm{ssGeo}(q, M, m), \mathrm{ssGeo}(pq, M, m), \mathrm{sdAL}(\mathrm{p}, \mathrm{q}, \mathrm{m}))$

F_{Exp} is a zero-temperature version ($=$ a tropicalization, an ultra-discretization) of F_{Ga}, namely $(+, \times)$-algebra is replaced by $(\min, +)$-algebra.

Known results

- $F_{\mathrm{GIG-Ga}} : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \, : \, F_{\mathrm{GIG-Ga}}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y} \right)$:
  Generalized inverse gaussian distribution
  (Matsumoto-Yor 2001(if part), Letac-Wesolowski 2000 (only if part))
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{GIG}(a,b,c), \mathrm{Ga}(a,b), \mathrm{GIG}(a,c,b), \mathrm{Ga}(a,c))$
- $F_{\mathrm{Be}} : (0,1)^2 \to (0,1)^2 \, : \, F_{\mathrm{Be}}(x,y) = \left(\frac{1-x}{1-xy}, 1 - xy \right)$: Beta distribution (Seshadri-Wesolowski 2003)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{Be}(a,b), \mathrm{Be}(a+b,c), \mathrm{Be}(c,b), \mathrm{Be}(c+b,a))$
- $F_{\mathrm{K-Ga,A}} : \mathbb{R}_+^2 \to \mathbb{R}_+ \times (0,1) \, : \, F_{\mathrm{K-Ga,A}}(x,y) = \left(x+y, \frac{1+\frac{1}{x+y}}{1+\frac{1}{x}} \right)$:
  Kummer distribution (Koudou-Vallois 2012, Piliszek and Wesolowski 2018)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{K}^{(2)}(a,b,c), \mathrm{Ga}(b,c), \mathrm{K}^{(2)}(a+b,-b,c), \mathrm{Be}(a,b))$

Known results

- $F_{\mathrm{GIG-Ga}} : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \,:\, F_{\mathrm{GIG-Ga}}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y}\right)$: Generalized inverse gaussian distribution (Matsumoto-Yor 2001(if part), Letac-Wesolowski 2000 (only if part))
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{GIG}(a,b,c), \mathrm{Ga}(a,b), \mathrm{GIG}(a,c,b), \mathrm{Ga}(a,c))$
- $F_{\mathrm{Be}} : (0,1)^2 \to (0,1)^2 \,:\, F_{\mathrm{Be}}(x,y) = \left(\frac{1-x}{1-xy}, 1-xy\right)$: Beta distribution (Seshadri-Wesolowski 2003)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{Be}(a,b), \mathrm{Be}(a+b,c), \mathrm{Be}(c,b), \mathrm{Be}(c+b,a))$
- $F_{\mathrm{K-Ga,A}} : \mathbb{R}_+^2 \to \mathbb{R}_+ \times (0,1) \,:\, F_{\mathrm{K-Ga,A}}(x,y) = \left(x+y, \frac{1+\frac{1}{x+y}}{1+\frac{1}{x}}\right)$: Kummer distribution (Koudou-Vallois 2012, Piliszek and Wesolowski 2018)
  $(\mu, \nu, \tilde{\mu}, \tilde{\nu}) = (\mathrm{K}^{(2)}(a,b,c), \mathrm{Ga}(b,c), \mathrm{K}^{(2)}(a+b,-b,c), \mathrm{Be}(a,b))$
- $F_{\mathrm{Be}}^{\delta}(x,y) = \left(\frac{1-xy}{1+(\delta-1)xy}, \; \frac{1-x}{1+(\delta-1)x}\frac{1+(\delta-1)xy}{1-xy}\right)$ for $\delta > 0$
- $F_{\mathrm{K-Ga},B}(x,y) = \left(\frac{y}{1+x}, \frac{x(1+x+y)}{1+x}\right)$

Remarks

- For $F_{\mathrm{Ga}}, F_{\mathrm{GIG-Ga}}, F_{\mathrm{Be}}, F_{\mathrm{K-Ga,A}}$, the positive symmetric $n \times n$ matrix versions also exist. Namely, $F : \mathrm{Sym}_+^n \times \mathrm{Sym}_+^n \to \mathrm{Sym}_+^n \times \mathrm{Sym}_+^n$. Some cases are even generalized to symmetric cones.
- They (including their generalized versions) are all known examples before we found new classes of examples originated from integrable systems.
- The results for F_{Ga} and F_{Be} were keys of the characterization of stationary $1+1$ dimensional lattice polymer models (Chaumont-Noack, 2017)
- All continuous examples have solutions with exactly three parameters, but there was no explanation for this coincidence.
- Free probability analogues for F_{N} (Nica 1996), F_{Ga} (K. Szpojankowski, 2015), $F_{\mathrm{GIG-Ga}}$ (K. Szpojankowski, 2017) are also known.

Answer to "Basic question"

- Q : For which F, the independence preserving property holds?

Theorem (Koudou and Vallois (2012))

Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a regular decreasing bijection and let $F : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ be $F(x, y) = (f(x + y), f(x) - f(x + y))$ (called Matsumoto-Yor type).
If F has the IP property with distributions having regular densities, then $F = F_{\mathrm{GIG-Ga}}, F_{\mathrm{Be}}^{\delta}$ or $F_{\mathrm{K-Ga,A}}$ up to the natural equivalence.

- Other than this result, nothing was studied in a unified way. Just specific examples were known.

Answer to "Basic question"

- Q : For such F, can we characterize all solutions $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ for $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$?
- A : Most of known examples where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ are an open interval of $\mathbb{R}$ and the function F is smooth, the complete characterization has been obtained. For other general cases, such as $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ are space of positive definite symmetric matrices, or the function F includes min function (zero-temperature version), the characterization becomes more complicated, and less are known.

Our results

Zero-temperature (ultra-discrete) version

- Except F_{N} and $F_{\mathrm{Exp}}(=$ zero-temp. version of F_{Ga}), all known examples have zero-temp. version, which have IP property too.
- Applying the zero-temp. version, we find a new stationary $1+1$ dimensional lattice "zero-temperature" polymer model

Subtraction-free expressions as $F : \mathbb{R}_+^2 \to \mathbb{R}_+^2$

If $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ are open intervals of $\mathbb{R}$, by a coordinate-wise change of variables, we can change the domains and codomains of F to $\mathbb{R}_+^2$. In particular, we have the following normalized expressions, and so can take the zero-temp. limit for all!

- $F^+_{\mathrm{Ga}}(x,y) = F_{\mathrm{Ga}}(x,y) = \left(x+y, \frac{x}{y}\right), \ (F^+_{\mathrm{Ga}})^{-1}(x,y) = \left(\frac{xy}{1+y}, \frac{x}{1+y}\right)$
- $F^{+,\delta}_{\mathrm{Be}}(x,y) = \left(\frac{1+x+y}{\delta xy}, \ \frac{1+x+y+\delta xy}{x(\delta+\delta x)}\right)$
- $F^+_{\mathrm{K-Ga},A}(x,y) = \left(x+y, \frac{x(x+y+1)}{y}\right), \ (F^+_{\mathrm{K-Ga},A})^{-1}(x,y) = \left(\frac{xy}{1+x+y}, \frac{x(1+x)}{1+x+y}\right)$
- $F^+_{\mathrm{K-Ga},B}(x,y) = F_{\mathrm{K-Ga},B}(x,y) = \left(\frac{y}{1+x}, \frac{x(1+x+y)}{1+x}\right)$
- $F^+_{\mathrm{Be}}(x,y) = F^{+,1}_{\mathrm{Be}}(x,y) = \left(\frac{1+x+y}{xy}, \ \frac{1+y}{x}\right)$
- $F^+_{\mathrm{GIG-Ga}} = \left(\frac{y}{1+xy}, \ x(1+xy)\right)$

Our results

New examples having the IP property

- Studying invariant measures of the discrete modified KdV equation, we find a new class of functions $F_{\mathrm{GIG}}^{\alpha,\beta}(x,y) = \left(y\frac{1+\beta xy}{1+\alpha xy},\ x\frac{1+\alpha xy}{1+\beta xy} \right) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ where $\alpha, \beta \geq 0$ having the IP property. The class was known as an example of Yang-Baxter maps.
- In a class of Yang-Baxter maps on $\mathbb{R}_+^2$, we find new classes of functions $H_I^{+,\alpha,\beta}, H_{II}^{+,\alpha,\beta}$ where $\alpha, \beta \geq 0$, having the IP property. (S-Uozumi, 2022)

Our results

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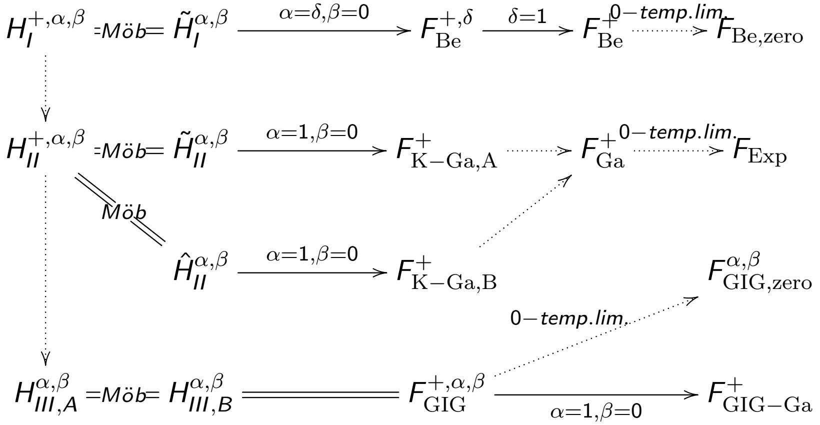
- In a class of Yang-Baxter maps on $\mathbb{R}_+^2$, we find new classes of functions $H_I^{+,\alpha,\beta}, H_{II}^{+,\alpha,\beta}$ where $\alpha, \beta \geq 0$, having the IP property. (S-Uozumi, 2022)

Relations between bijections having the IP property (S-Uozumi, 2022)

Unified relation between known examples



All known maps except F_{N} are understood in a unified manner!

Discrete KdV equation (1 + 1 dimensional lattice model)

- $n \in \mathbb{Z}$: space variable, $t \in \mathbb{Z}$: time variable
- $0 < \delta < 1$: model parameter, $x_n^t > 0$

Discrete KdV equation :

$$\frac{1}{x_{n+1}^{t+1}} - \frac{1}{x_n^t} = \delta(x_{n+1}^t - x_n^{t+1}).$$

Invariant measures?
How to define the dynamics for a given initial configuration $(x_n^0)_{n \in \mathbb{Z}}$?

Let (formally) $y_n^t := \prod_{m=-\infty}^{n} \frac{x_m^t}{x_m^{t+1}}$. Then, the relation is rewritten as

$$F_{\mathrm{dK}}^{(\delta)}(x_n^{t+1}, y_n^t) = (x_n^t, y_{n-1}^t)$$

where $F_{\mathrm{dK}}^{(\delta)}(x, y) = \left(\frac{y}{1+\delta xy}, x(1+\delta xy)\right)$, which is an involution.

$1+1$ dimensional deterministic lattice dynamics

KdV-type locally-defined dynamics

- $\mathcal{X}_0, \mathcal{Y}_0$: sets
- $F : \mathcal{X}_0 \times \mathcal{Y}_0 \to \mathcal{X}_0 \times \mathcal{Y}_0$: involution, namely $F = F^{-1}$
- $\mathcal{X} := \mathcal{X}_0^{\mathbb{Z}}$

Initial value problem

For a given initial value $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{X}$, $(x_n^t, y_n^t)_{n,t\in\mathbb{Z}}$ is a solution of the initial value problem to the locally-defined dynamics F if

$$\begin{cases} (x_n^{t+1}, y_n^t) = F(x_n^t, y_{n-1}^t) & n, t \in \mathbb{Z} \\ x_n^0 = x_n. \end{cases}$$

If for $\mathbf{x} \in \mathcal{X}$, the solution exists uniquely, then we can define the one-time step dynamics $\mathbf{x} \to T\mathbf{x}$ as $(x_n) \to (x_n^1)$.

$\mathcal{X}^* := \{\mathbf{x} \in \mathcal{X} \ : \ \exists! \text{ solution of the initial value problem for } \mathbf{x}\}$.

I.i.d. invariant measures

Suppose proper measurability related conditions on $\mathcal{X}_0, \mathcal{Y}_0$ and F to study invariant measures.

Theorem (Croydon-S, 2021)

Let μ be a probability measure on $\mathcal{X}_0$ satisfying $\mu^{\mathbb{Z}}(\mathcal{X}^*) = 1$. Then, $\mu^{\mathbb{Z}} = T\mu^{\mathbb{Z}}$ (i.e. $\mu^{\mathbb{Z}}$ is invariant)
$\Leftrightarrow \exists \nu$: a probability measure on $\mathcal{Y}_0$ such that

$$F(\mu \times \nu) = \mu \times \nu.$$

Moreover, if it is the case, $(x_n^t)_n \sim \mu^{\mathbb{Z}}$ for any $t \in \mathbb{Z}$ and $(y_n^t)_t \sim \nu^{\mathbb{Z}}$ for any $n \in \mathbb{Z}$ when $(x_n^0)_n \sim \mu^{\mathbb{Z}}$. (Burke property)

In particular, if the dynamics has an i.i.d. non-dirac invariant measure, then F must have the IP property.

Examples

modified discrete KdV equation

$$F^{\alpha,\beta}_{\mathrm{mdKdV}}(x,y) = \left(\frac{y(1+\beta xy)}{1+\alpha xy}, \frac{x(1+\alpha xy)}{1+\beta xy}\right),$$
$$F^{\delta,0}_{\mathrm{mdKdV}} = F^{(\delta)}_{\mathrm{dK}}$$

modified ultra-discrete KdV equation

$$F^{J,K}_{\mathrm{mudKdV}}(x,y) = (y - \max\{x+y-J,0\} + \max\{x+y-K,0\},$$
$$x - \max\{x+y-K,0\} + \max\{x+y-J,0\}),$$
$$F^{L,\infty}_{\mathrm{mudKdV}} = F^{(L)}_{\mathrm{udK}}(x,y) = (\min\{L-x,y\}, x+y-\min\{L-x,y\})$$

Independence preserving property for $F^{J,K}_{\mathrm{mudKdV}}$ and $F^{\alpha,\beta}_{\mathrm{mdKdV}}$

We study the solution of $F^{\alpha,\beta}_{\mathrm{mdKdV}}(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$ and $F^{J,K}_{\mathrm{mudKdV}}(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$:

$F^{\alpha,\beta}_{\mathrm{mdKdV}}$ (Croydon-S (if part), 2020, Bao-Noack, 2021, Letac-Wesolowski, 2022 (only if part)) : Generalize inverse Gaussian distribution :

$$(\mu, \nu, \tilde{\mu}, \tilde{\nu}) \\ = (\mathrm{GIG}(-\lambda, \alpha a, b), \mathrm{GIG}(-\lambda, \beta b, a), \mathrm{GIG}(-\lambda, \alpha b, a), \mathrm{GIG}(-\lambda, \beta a, b))$$

$F^{J,K}_{\mathrm{mudKdV}}$ (Croydon-S, 2020, Bao-Noack, 2021 (not completely characterized)) : Truncated Exponential / Truncated (bipartite) Geometric distribution

From these results, we obtain i.i.d. invariant measures for ultra-discrete KdV and discrete KdV equation.

Remarks

- To apply our general theorem, we need to know $\mathcal{X}^*$. This is not a simple task, but we also have a general strategy where we use generalized Pitman's transform (Croydon-S-Tsujimoto, 2022). We succeed to check that $\mu^{\mathbb{Z}}(\mathcal{X}^*) = 1$ for the discrete KdV, the ultra-discrete KdV and a special class of the modified ultra-discrete KdV, but not for the modified discrete KdV.
- $F^{(\delta)}_{\mathrm{dKdV}}$ coincides with $F_{\mathrm{GIG-Ga}}$ by a change of variable.
- The dynamics $F^{J,K}_{\mathrm{mudKdV}}$ corresponds to the box-ball system with the box capacity J and the carrier capacity K.
- To understand Generalized Gibbs ensembles (GGE), it is crucial to understand invariant measures of discrete integrable models.
- $F^{\alpha,\beta}_{\mathrm{mdKdV}}$ satisfies the Yang-Baxter relation.

Discrete Toda equation $(1+1$ dimensional lattice model$)$

- $n \in \mathbb{Z}$: space variable, $t \in \mathbb{Z}$: time variable
- $I_n^t > 0, V_n^t > 0$

Discrete Toda equation :

$$\begin{cases} I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1} \\ V_n^{t+1} = \frac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases}$$

Invariant measures? How to define the dynamics??

Let (formally) $y_n^t := \frac{\prod_{j=-\infty}^{n} I_j^t}{\prod_{j=-\infty}^{n-1} I_j^{t+1}}$. Then, the relation is rewritten as

$$(I_n^{t+1}, V_n^{t+1}, y_n^t) = F_{\mathrm{dT}}(I_{n+1}^t, V_n^t, y_{n-1}^t)$$

where $F_{\mathrm{dT}}(a,b,c) = \left(b+c, \frac{ab}{b+c}, \frac{ac}{b+c}\right)$, which is an involution.

F_{dT} is "decomposed" into $F_{\mathrm{dT}}^*(x,y) = \left(x+y, \frac{x}{x+y}\right)$ and its inverse.

$1+1$ dimensional lattice dynamics

Toda-type locally-defined dynamics

- $\mathcal{X}_0, \tilde{\mathcal{X}}_0, \mathcal{Y}_0, \tilde{\mathcal{Y}}_0$: sets
- $F^* : \mathcal{X}_0 \times \mathcal{Y}_0 \to \tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Y}}_0$: bijection
- $F_{2n} := F^*, \quad F_{2n+1} := (F^*)^{-1}$
- $\mathcal{X} := (\mathcal{X}_0 \times \tilde{\mathcal{X}}_0)^{\mathbb{Z}}$

Initial value problem

For a given initial value $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{X}$, $(x_n^t, y_n^t)_{n,t\in\mathbb{Z}}$ is a solution of the initial value problem to the locally-defined dynamics F if

$$\begin{cases} (x_{n-1}^{t+1}, y_n^t) = F_n(x_n^t, y_{n-1}^t) \quad n, t \in \mathbb{Z} \\ x_n^0 = x_n. \end{cases}$$

$\mathcal{X}^* := \{\mathbf{x} \in \mathcal{X} \ : \ \exists!$ solution of the initial value problem for $\mathbf{x}\}$.

Alternate i.i.d. type invariant measures

Suppose proper measurability related conditions.

Theorem (Croydon-S, 2021)

Let $\mu, \tilde{\mu}$ *be probability measures on on* $\mathcal{X}_0, \tilde{\mathcal{X}}_0$ *satisfying* $(\mu \times \tilde{\mu})^{\mathbb{Z}}(\mathcal{X}^*) = 1$.
Then, $(\mu \times \tilde{\mu})^{\mathbb{Z}} = T(\mu \times \tilde{\mu})^{\mathbb{Z}}$ *(i.e.* $(\mu \times \tilde{\mu})^{\mathbb{Z}}$ *is invariant)*
$\Leftrightarrow \exists \nu, \tilde{\nu}$ *: probability measures on* $\mathcal{Y}_0, \tilde{\mathcal{Y}}_0$ *such that*

$$F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}.$$

Burke property holds too.

In particular, if the dynamics has an alternate i.i.d. type non-dirac invariant measure, then F^* must have the IP property.

Examples

discrete Toda equation

$$F^*_{\mathrm{dT}} = F_{\mathrm{Ga}}$$

ultra-discrete Toda equation

$$F^*_{\mathrm{udT}} = F_{\mathrm{Exp}}$$

By the classical characterization results, the alternate i.i.d. type invariant measures for the discrete Toda and the ultra-discrete Toda equations are completely characterized.

$(1+1)$-dimensional Directed random polymer model with edge weights

Directed random polymer models with edge weights :

- $(X_{n,m})_{n,m\in\mathbb{Z}_+}$: i.i.d. $(0,\infty)$-valued random variables
- $u:(0,\infty)\to(0,\infty)$
- $v:(0,\infty)\to(0,\infty)$

where $\mathbb{Z}_+ := \{0,1,\dots\}$.

The partition function

$$Z_{n,m} = \sum_{\pi:(0,0)\to(n,m)} \left\{ \prod_{e\in\pi} Y_e \right\}, \qquad \forall n,m\in\mathbb{Z}_+,$$

where the sum is taken over up-right paths $\pi=(e_1,e_2,\dots,e_{n+m})$ from $(0,0)$ to (n,m) on $\mathbb{Z}_+^2$, and the edge weights are defined by setting

$$Y_e := \begin{cases} u(X_{k,\ell}), & \text{if } e=((k-1,\ell),(k,\ell)), \\ v(X_{k,\ell}), & \text{if } e=((k,\ell-1),(k,\ell)). \end{cases}$$

Directed random polymer model with edge weights

The recursive equation for the partition function:

$$Z_{n,m} = u(X_{n,m})Z_{n-1,m} + v(X_{n,m})Z_{n,m-1}.$$

By setting

$$U_{n,m} := Z_{n,m}/Z_{n-1,m}, \qquad V_{n,m} := Z_{n,m}/Z_{n,m-1},$$

the recursive equation can be rewritten as

$$R_{RPe}(X_{n,m}, U_{n,m-1}, V_{n-1,m}) = (U_{n,m}, V_{n,m}),$$

where

$$R_{RPe}(a,b,c) = \left(u(a) + v(a)\frac{b}{c}, u(a)\frac{c}{b} + v(a)\right).$$

Directed random polymer model with edge weights

The recursive equation for the partition function:

$$Z_{n,m} = u(X_{n,m})Z_{n-1,m} + v(X_{n,m})Z_{n,m-1}.$$

By setting

$$U_{n,m} := Z_{n,m}/Z_{n-1,m}, \qquad V_{n,m} := Z_{n,m}/Z_{n,m-1},$$

the recursive equation can be rewritten as

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where

$$R_{RPe}(a,b,c) = \left(u(a) + v(a)\frac{b}{c}, u(a)\frac{c}{b} + v(a)\right).$$

If $u(x) = v(x) = x$, then the model reduces to a directed random polymer model with site weights, namely the partition function can be written

$$Z_{n,m} = \sum_{\pi:(0,0)\to(n,m)} \left\{ \prod_{(k,\ell)\in\pi\setminus\{(0,0)\}} X_{k,\ell} \right\}.$$

$(1+1)$-dimensional stochastic lattice models

$\mathbb{Z}^2$ (whole lattice) version (formal)

- $(X_{n,m})_{n,m\in\mathbb{Z}^2}$: i.i.d. random variables
- $(Z_{n,m})_{n,m\in\mathbb{Z}^2}$: the partition function (formally) determined by $(X_{n,m})$
- $(U_{n,m})_{n,m\in\mathbb{Z}^2}$: the 'derivative' (in a suitable sense) of $(Z_{n,m})$ in the first-coordinate directions
- $(V_{n,m})_{n,m\in\mathbb{Z}^2}$: the 'derivative' (in a suitable sense) of $(Z_{n,m})$ in the second-coordinate directions

Typically one has a recursive relation of the form:

$$R\left(X_{n,m}, U_{n,m-1}, V_{n-1,m}\right) = \left(U_{n,m}, V_{n,m}\right), \tag{1}$$

where $R : J_1 \times I_1 \times I_2 \to I_1 \times I_2$ for some subsets $J_1, I_1, I_2 \subseteq \mathbb{R}$ (usually intervals or discrete subsets).

Stationary solutions

If there exists a triplet of probability measures $(\tilde{\mu}, \mu, \nu)$ on J_1, I_1 and I_2 such that

$$R(\tilde{\mu} \times \mu \times \nu) = \mu \times \nu, \tag{2}$$

then we say that the stochastic lattice model has a stationary solution with

$$X_{n,m} \sim \tilde{\mu}, \quad U_{n,m} \sim \mu, \quad V_{n,m} \sim \nu.$$

In fact, we can construct 'upper-right corner'-version and then by using its translation, we can construct 'stationary version'.

'upper-right corner'-version

- $(X_{n,m})_{n,m\in\mathbb{N}}$: i.i.d. random variables
- $(U_{n,0})_{n\in\mathbb{N}}$: i.i.d. random variables
- $(V_{0,n})_{n\in\mathbb{N}}$: i.i.d. random variables
- $(U_{n,m})_{n,m\in\mathbb{N}}$ and $(V_{n,m})_{n,m\in\mathbb{N}}$: Determined by the recursive relation (1).

Characterization of stationary solutions

"Problem : Find a stationary model"
is reduced to the following :
Problem : Find a triplet of probability measures $(\tilde{\mu}, \mu, \nu)$ such that

$$R\left(\tilde{\mu} \times \mu \times \nu\right) = \mu \times \nu.$$

Moreover, Chaumont and Noack reduced this problem to find a solution of $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$ for a properly chosen F. In particular, for their study, F_{Ga} and F_{Be} appear.

Characterization of stationary solutions by H. Chaumont and C. Noack

Suppose

- $u(x) = x$,

and some further technical assumptions on v and also measures $\mu, \nu, \tilde{\mu}$.

Theorem (H. Chaumont and C. Noack (2018))

Under the assumption, R_{PRe} has a triplet of probability measures $(\tilde{\mu}, \mu, \nu)$ satisfying (2) if and only if $v(x) = \alpha + \beta x$ for some $\alpha, \beta \in \mathbb{R}$ such that $\max\{\alpha, \beta\} > 0$. By making simple changes of variables, these can be reduced to the four cases with (α, β) being given by $(0, 1), (1, 0), (-1, 1)$ or $(1, -1)$. Moreover, the solutions are completely characterized explicitly, and the distribution of $X_{n,m}$ is inverse-gamma, gamma, inverse-beta, and beta respectively, and each model has three parameters.

Zero-temperature limit of the model

- $(X_{n,m})_{n,m\in\mathbb{Z}_+}$: i.i.d. random variables
- $u, v : \mathbb{R} \to \mathbb{R}$

The partition function

$$Z_{n,m} = \min_{\pi:(0,0)\to(n,m)} \left\{ \sum_{e\in\pi} Y_e \right\}, \qquad \forall n, m \in \mathbb{Z}_+,$$

where

$$Y_e := \begin{cases} u(X_{k,\ell}), & \text{if } e = ((k-1,\ell),(k,\ell)), \\ v(X_{k,\ell}), & \text{if } e = ((k,\ell-1),(k,\ell)). \end{cases}$$

The recursive equation for the partition function:

$$Z_{n,m} = \min\{u(X_{n,m}) + Z_{n-1,m}, v(X_{n,m}) + Z_{n,m-1}\}$$

By setting $U_{n,m} := Z_{n,m} - Z_{n-1,m}$, $V_{n,m} := Z_{n,m} - Z_{n,m-1}$, the recursive equation can be rewritten as

$$R^{zero}_{RPe}(a,b,c) = \left(\min\{u(a), v(a) + b - c\}, \min\{u(a) + c - b, v(a)\}\right).$$

Special cases

$R^{zero}_{(\alpha,\beta)}$: a proper zero-temperature limit of R_{RPe} with $u(x) = x$ and $v(x) = \alpha + \beta x$. They are known in literature as :

- $(\alpha, \beta) = (0, 1)$: Directed last passage percolation (LPP) with exponential/geometric waiting times ($u(x) = v(x) = x$)
- $(\alpha, \beta) = (1, 0)$: Directed first passage percolation (FPP) with exponential/geometric waiting times ($u(x) = x, v(x) = 0$)
- $(\alpha, \beta) = (1, 1)(\text{``} = \text{''}(-1, 1))$: Bernoulli-exponential/geometric polymer ($u(x) = x, v(x) = \min\{x, 0\}$)
- $(\alpha, \beta) = (1, -1)$: Bernoulli-exponential/geometric FPP, introduced as a zero-temperature limit of the β-RWRE, also called the river delta model ($u(x) = -\min\{x, 0\}, v(x) = \max\{x, 0\}$)

Explicit stationary solutions : continuous measures

Theorem (Croydon-S, 2022)

For each $R^{zero}_{(\alpha,\beta)}$ with $(\alpha,\beta)=(0,1),(1,0),(1,1)$ or $(1,-1)$, we explicitly obtained a class of triplet $(\tilde{\mu},\mu,\nu)$ satisfying (2). Each class of measures consists of continuous measures (which may have an atom at 0) with three parameters and discrete measures with four parameters.

- Directed LPP $R^{zero}_{(0,1)}$ (site weights)
  $(\tilde{\mu},\mu,\nu)=(-sExp(\rho+\sigma,\tau),-sExp(\rho,\tau),-sExp(\sigma,\tau))$
- Directed FPP $R^{zero}_{(1,0)}$
  $(\tilde{\mu},\mu,\nu)=(sExp(\sigma,\tau),sExp(\rho+\sigma,\tau),\min\{AL(\sigma,\rho),0\})$
- Bernoulli-exponential/geometric polymer $R^{zero}_{(1,1)}$
  $(\tilde{\mu},\mu,\nu)=(AL(\rho,\sigma+\tau),AL(\rho+\sigma,\tau),\min\{AL(\rho,\sigma),0\}))$
- Bernoulli-exponential/geometric FPP $\tilde{R}^{zero}_{(1,-1)}$
  $(\tilde{\mu},\mu,\nu)=(AL(\tau,\sigma),\max\{AL(\rho+\sigma,\tau),0\},\min\{AL(\sigma,\rho),0\})$

- Discrete versions are obtained by $sEXP\to ssGeo$, $AL\to dAL$

Remarks

- The explicit solution for the river delta model was not known.
- Key of the proof is to introduce the zero-temperature version of F_{Be} through F^{+}_{Be}, and also connect R and F properly. Some new aspects of zero-temperature version were found.
- The proof explains how the Bernoulli-exponential distribution appear naturally in these models.

- Recently, the matrix-valued versions of stationary random polymer models related have been studied by O' Connell. The stationarity is explained by the IP property of the matrix version of F_{Ga}.

1. Independence preserving property

2. Classical integrable systems

3. Stochastic integrable models

4. Yang-Baxter map

Yang-Baxter map

A bijection $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is called a Yang-Baxter map if

$$F_{12} \circ F_{13} \circ F_{23} = F_{23} \circ F_{13} \circ F_{12}$$

where

$$F_{ij} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X} \times \mathcal{X}$$

acts on the i-th and j-th factors.
Namely, denoting $(u(x,y), v(x,y)) = F(x,y)$,

$$\begin{aligned}
F_{12}(x_1, x_2, x_3) &= (u(x_1, x_2), v(x_1, x_2), x_3), \\
F_{13}(x_1, x_2, x_3) &= (u(x_1, x_3), x_2, v(x_1, x_3)), \\
F_{23}(x_1, x_2, x_3) &= (x_1, u(x_2, x_3), v(x_2, x_3)).
\end{aligned}$$

Introduced by Drinfeld as the “set-theoretical” Yang-Baxter equation in 1992.

Parameter dependent Yang-Baxter map

A family of bijections $F(\alpha,\beta): \mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ with parameters α,β in a certain set of parameters Θ, they are Yang-Baxter maps if

$$F_{12}(\lambda_1,\lambda_2)\circ F_{13}(\lambda_1,\lambda_3)\circ F_{23}(\lambda_2,\lambda_3) = F_{23}(\lambda_2,\lambda_3)\circ F_{13}(\lambda_1,\lambda_3)\circ F_{12}(\lambda_1,\lambda_2)$$

holds for any parameters λ_1,λ_2 and $\lambda_3\in\Theta$.

By replacing $\mathcal{X}$ with $\mathcal{X}\times\Theta$ and considering

$$\tilde{F}((x,\alpha),(y,\beta)) := \left((u(\alpha,\beta)(x,y),\alpha\right), \left(v(\alpha,\beta)(x,y),\beta)\right)$$

where $F(\alpha,\beta)(x,y) = (u(\alpha,b)(x,y), v(\alpha,\beta)(x,y))$, we obtain a (parameter-independent) Yang-baxter map $\tilde{F}$.

Questions about the relation between the IP property and the integrability

Recall that $(F^{\alpha,\beta}_{\mathrm{mdKdV}})_{\alpha,\beta}$ satisfies the parameter dependent Yang-Baxter relation.

- Do other bijections having IP property satisfy Yang-Baxter relation in a reasonable sense?
- Do all integrable lattice dynamics have i.i.d. invariant measures?
- Is there any relation between Yang-Baxter maps and the independence preserving property?

Classification of quadrirational Yang-Baxter maps

- For $\mathcal{X} = \mathbb{CP}^1$, some classification results of Yang-Baxter maps were known.
- $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1, \quad (x, y) \mapsto (u(x, y), v(x, y))$: birational if F and F^{-1} are both rational functions.
- Companion map $\bar{F}$ of F is defined as $\bar{F}(u, y) = (x, v)$ for $(u, v) = F(x, y)$ if it is well-defined.
- F : quadrirational if its companion map $\bar{F}$ is well-defined and F and $\bar{F}$ are both birational functions. (Adler et al. 2004)

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- Companion map $\bar{F}$ of F is defined as $\bar{F}(u, y) = (x, v)$ for $(u, v) = F(x, y)$ if it is well-defined.
- F : quadrirational if its companion map $\bar{F}$ is well-defined and F and $\bar{F}$ are both birational functions. (Adler et al. 2004)

Theorem (Adler, Bobenko, Suris (2004))

Any quadrirational map $F(x, y) = (u(x, y), v(x, y))$ has the form:

$$u(x, y) = \frac{a(y)x + b(y)}{c(y)x + d(y)}, \quad v(x, y) = \frac{A(x)y + B(x)}{C(x)y + D(x)}$$

where $a(y), \ldots, d(y)$ are polynomials in y and $A(x), \ldots, D(x)$ are polynomials in x, whose degrees are all less than or equal to two.

Classification of quadrirational Yang-Baxter maps

There exist three subclasses of such maps, denoted by pair of numbers as $[1:1], [1:2]$ and $[2:2]$ depending on the highest degrees of the coefficients of the polynomials for x and y.

The most rich and interesting subclass is $[2:2]$.

Theorem (Adler, Bobenko, Suris (2004))

In the subclass $[2:2]$ of quadrirational maps, up to the coordinate-wise Möbius transformations, there are only five families of quadrirational maps $F_I = (F_I^{\alpha,\beta}), F_{II} = (F_{II}^{\alpha,\beta}), \ldots, F_V = (F_V^{\alpha,\beta})$ where all have parameters $\alpha, \beta \in \mathbb{C}$.

Remarkably, all of these five canonical representative maps $F_I = (F_I^{\alpha,\beta}), F_{II} = (F_{II}^{\alpha,\beta}), \ldots, F_V = (F_V^{\alpha,\beta})$ are Yang-Baxter maps!

Which map gives $F : \mathbb{R}_+^2 \to \mathbb{R}_+^2$?

Papageorgiou et al. (2010) pointed out that F_I, F_{II} and F_{III} have subtraction-free forms.

$$H_I^+(x,y) = \left(\frac{y}{\alpha} \frac{\beta + \alpha x + \beta y + \alpha\beta xy}{1 + x + y + \beta xy},\ \frac{x}{\beta} \frac{\alpha + \alpha x + \beta y + \alpha\beta xy}{1 + x + y + \alpha xy} \right)$$
$$H_{II}^+(x,y) = \left(\frac{y}{\alpha} \frac{\beta + \alpha x + \beta y}{1 + x + y},\ \frac{x}{\beta} \frac{\alpha + \alpha x + \beta y}{1 + x + y} \right)$$
$$H_{III,A}(x,y) = \left(\frac{y}{\alpha} \frac{\alpha x + \beta y}{x + y},\ \frac{x}{\beta} \frac{\alpha x + \beta y}{x + y} \right)$$

Natural restriction from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{R}_+ \times \mathbb{R}_+$ exists.
$H_{III,A}$ is equivalent to F_{mdKdV} up to a coordinate-wise change of variables.

IP property for quadrirational Yang-Baxter maps

Theorem (S-Uozumi(2022))

For the following distributions (X, Y), *each* F *has the IP property.*

(i) $F = H_I^{+,\alpha,\beta}$. *For* $\lambda \in \mathbb{R}$, $a, b > 0$, $-\min\{a, b\} < \frac{\lambda}{2} < \min\{a, b\}$,

$$X \sim \mathrm{Be}'(\lambda, a, b\ ;\ \alpha, 1), \quad Y \sim \mathrm{Be}'(-\lambda, a, b\ ;\ \beta, 1),$$
$$U \sim \mathrm{Be}'(-\lambda, a, b\ ;\ \alpha, 1), \quad V \sim \mathrm{Be}'(\lambda, a, b\ ;\ \beta, 1)$$

(ii) $F = H_{II}^{+,\alpha,\beta}$. *For* $\lambda \in \mathbb{R}$, $a, b > 0$, $-b < \frac{\lambda}{2} < b$,

$$X \sim \mathrm{K}(\lambda, a, b\ ;\ \alpha, 1), \quad Y \sim \mathrm{K}(-\lambda, a, b\ ;\ \beta, 1),$$
$$U \sim \mathrm{K}(-\lambda, a, b\ ;\ \alpha, 1), \quad V \sim \mathrm{K}(\lambda, a, b\ ;\ \beta, 1)$$

(iii) $F = H_{III,A}^{\alpha,\beta}$. *For* $\lambda \in \mathbb{R}$, $a, b > 0$,

$$X \sim \mathrm{GIG}(\lambda, a, b\ ;\ \alpha, 1), \quad Y \sim \mathrm{GIG}(-\lambda, a, b\ ;\ \beta, 1),$$
$$U \sim \mathrm{GIG}(-\lambda, a, b\ ;\ \alpha, 1), \quad V \sim \mathrm{GIG}(\lambda, a, b\ ;\ \beta, 1).$$

Probability distributions

Generalized Beta prime distribution (p, q) For $\lambda, a, b \in \mathbb{R}$, $-b < \frac{\lambda}{2} < a$, the *Generalized Beta prime* distribution $\mathrm{Be}'(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} (1 + px)^{-a-\frac{\lambda}{2}} (1 + qx^{-1})^{-b+\frac{\lambda}{2}}, \qquad x \in \mathbb{R}_+.$$

Kummer distribution of Type 2 (p, q) For $\lambda, b \in \mathbb{R}$, $a > 0$, $-b < \frac{\lambda}{2}$, the *Kummer* distribution of Type 2 $\mathrm{K}(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-apx} (1 + qx^{-1})^{-b+\frac{\lambda}{2}}, \qquad x \in \mathbb{R}_+.$$

Generalized inverse Gaussian distribution (p, q) For $\lambda \in \mathbb{R}$, $a, b > 0$, the *generalized inverse Gaussian* distribution $\mathrm{GIG}(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-apx} e^{-bqx^{-1}}, \qquad x \in \mathbb{R}_+.$$

Scaling relations

Yang-Baxter maps

$$(i)\ \lim_{\epsilon\downarrow 0} H_I^{+,\epsilon\alpha,\epsilon\beta} = H_{II}^{+,\alpha,\beta}$$

$$(ii)\ \lim_{\epsilon\downarrow 0} (\theta_\epsilon \times \theta_\epsilon) \circ H_{II}^{+,\alpha,\beta} \circ (\theta_\epsilon \times \theta_\epsilon)^{-1} = H_{III,A}^{\alpha,\beta}$$

where $\theta_\epsilon(x) = \epsilon x : \mathbb{R}_+ \to \mathbb{R}_+$.

Probability distributions

$$(i)\ \lim_{\epsilon\downarrow 0} (1+\epsilon p x)^{-\frac{a}{\epsilon}-\frac{\lambda}{2}} = e^{-apx}$$

$$(ii)\ \lim_{\epsilon\downarrow 0} \mathrm{Be}'(\lambda, \frac{a}{\epsilon}, b; \epsilon p, q) = \mathrm{K}(\lambda, a, b; p, q)$$

$$(iii)\ \lim_{\epsilon\downarrow 0} (1+\epsilon q x^{-1})^{-\frac{b}{\epsilon}-\frac{\lambda}{2}} = e^{-bqx^{-1}}$$

$$(iv)\ \lim_{\epsilon\downarrow 0} \mathrm{K}(\lambda, a, \frac{b}{\epsilon}; p, \epsilon q) = \mathrm{GIG}(\lambda, a, b; p, q)$$

Hence, claims (ii) and (iii) of Theorem follows from the claim (i) of Theorem.

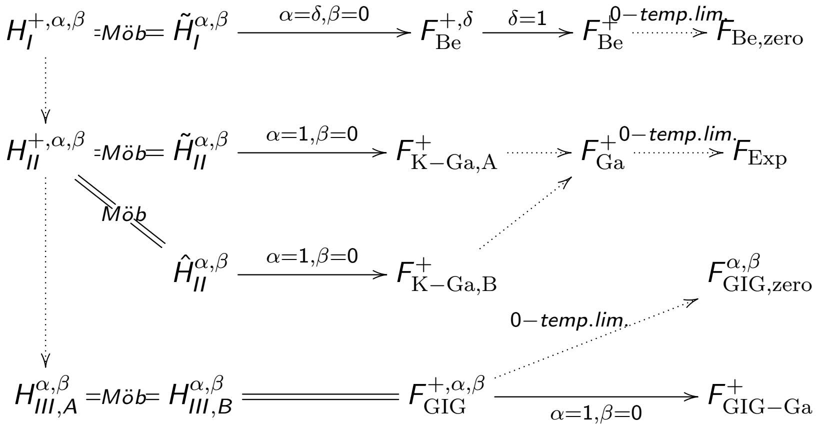
Special cases

The bijections $F^{+,\delta}_{\mathrm{Be}}$, $F^{+}_{\mathrm{K-Ga},A}$, $F^{+}_{\mathrm{K-Ga},B}$, F^{+}_{Ga} and $F^{+,\alpha,\beta}_{\mathrm{GIG}}$ are obtained from one of $H^{+,\alpha,\beta}_{I}$, $H^{+,\alpha,\beta}_{II}$ and $H^{\alpha,\beta}_{III,A}$ by Möbius transformations and singular limits.

Theorem

(i) $F^{+,\delta}_{\mathrm{Be}} = \tilde{H}^{\delta,0}_{I}$ *where* $\tilde{H}^{\alpha,\beta}_{I} = ((I \circ \theta_\alpha) \times (I \circ \theta_\beta)) \circ H^{+,\alpha,\beta}_{I}$.
(ii) $F^{+}_{\mathrm{K-Ga},A} = \tilde{H}^{1,0}_{II}$ *where* $\tilde{H}^{\alpha,\beta}_{II} = H^{+,\alpha,\beta}_{II} \circ \left(\theta_{\alpha^{-1}} \times \theta_{\beta^{-1}}\right)$.
(iii) $F^{+}_{\mathrm{K-Ga},B} = \hat{H}^{1,0}_{II}$ *where* $\hat{H}^{\alpha,\beta}_{II} = \left(\theta_{\alpha^{-1}} \times \theta_{\beta^{-1}}\right) \circ H^{+,\frac{1}{\alpha},\frac{1}{\beta}}_{II} \circ (\theta_\alpha \times \theta_\beta)$.
(iv) $F^{+}_{\mathrm{Ga}} = \tilde{F}^{0}_{\mathrm{K-Ga},A}$ *where* $\tilde{F}^{\epsilon}_{\mathrm{K-Ga},A} = (\theta_{\epsilon^{-1}} \times I_d) \circ F^{+}_{\mathrm{K-Ga},A} \circ (\theta_\epsilon \times \theta_\epsilon)$.
(v) $F^{+}_{\mathrm{Ga}} = \tilde{F}^{0}_{\mathrm{K-Ga},B}$ *where* $\tilde{F}^{\epsilon}_{\mathrm{K-Ga},B} = \pi \circ (I \times \theta_\epsilon) \circ F^{+}_{\mathrm{K-Ga},B} \circ (\theta_{\epsilon^{-1}} \times \theta_{\epsilon^{-1}})$.
(vi) $F^{+,\alpha,\beta}_{\mathrm{GIG}} = ((I \circ \theta_\alpha) \times I_d) \circ H^{\alpha,\beta}_{III,A} \circ (I_d \times (I \circ \theta_\beta))$
where $I(x) = \frac{1}{x}$, $I_d(x) = x$.

Relations between bijections having the IP property



All known maps except F_{N} on the product of open intervals of $\mathbb{R}$ are understood in a unified manner! These relations and the IP property for H_I^{+} recover all the IP property (if part) for all bijections with continuous distributions except F_{N}.

Remarks

- The IP property for H_{II}^{+} was independently found by Wesolowski and Koudou earlier than us. Moreover, they proved the "only if" part, namely the characterization of solutions (Bernoulli, 2025).
- The IP property for H_{I}^{+} was completely new. The "only if" part, namely the characterization of solutions was recently proved by Kolodziejek, Letac, Piccioni and Wesolowski (arXiv:2501.17007).
- There is no complete classification of Yang-Baxter maps nor bijections with IP property, so there is nothing for sure, but it seems there might be some relation.

Ongoing and future works

- We also obtained the ultra-discrete version (zero-temperature version) of the IP property for three quadrirational maps (joint work with Kondou and Nakajima). Is there any geometric way to understand it?
- Can we construct a parameter-dependent Yang-Baxter map on the positive symmetric matrices of symmetric cones by using the matrix version of bijections having the IP property?
- Are there any classical/stochastic integrable models associated to other bijections having the IP property?
- Is there any direct relation between the Yang-Baxter maps and bijections having the IP property?