Macroscopic scaling limits for the box-ball system

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Base on joint works with D. Croydon, M. Mucciconi, S. Olla, T.Sasamoto and H.Suda

Goal

Goal: To provide a mathematically rigorous foundation for Generalized Hydrodynamics (GHD)!

- GHD is a hydrodynamic theory for (one-dimensional) many body integrable systems.
- GHD is expected to be universally applicable to classical systems, quantum systems, field theories, spin systems, cellular automata, etc.
- There are very few rigorous results deriving the "GHD equations" from concrete microscopic models via the space-time scaling limits

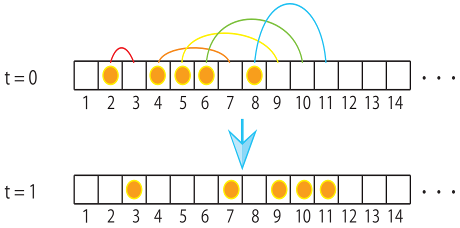
The box-ball system (BBS) is simple enough to be mathematically tractable, yet has a rich structure for studying the macroscopic behavior of many body integrable systems.

0. Box-ball system

Box-Ball System (BBS)

Def 1

- Every ball moves exactly once in each evolution time step
- The leftmost ball moves first and the next leftmost ball moves next and so on...
- Each ball moves to its nearest right vacant box



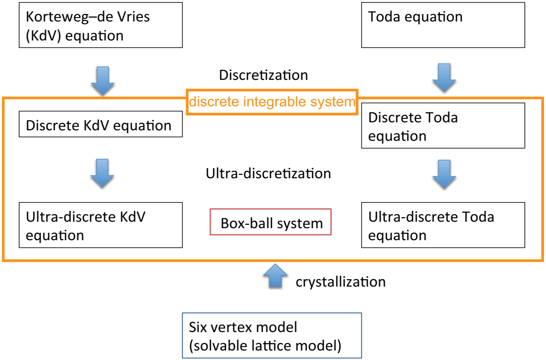
Notation

- $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$: Ball configuration
- $\Omega_{\mathrm{finite}} := \{\eta \in \{0,1\}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} \eta_n < \infty\}$: Set of all configurations with finite number of balls
- BBS dynamics map : $\mathcal{T} : \Omega_{\mathrm{finite}} \to \Omega_{\mathrm{finite}}$

Dynamics (Equation of motion):

$$\mathcal{T}\eta_n = \min\{1 - \eta_n, \sum_{m<n} (\eta_m - \mathcal{T}\eta_m)\}$$

Integrable systems around BBS



Ud-KdV equation : Euler representation of BBS

Ud-Toda equation : Lagrange representation of BBS

KdV equation and Toda lattice

- KdV equation : PDE on $\mathbb{R}$

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \qquad x \in \mathbb{R}$$

- Toda lattice : One-dimensional chain of oscillators with potential function $V(r) = \exp(-r) + r - 1$.

$$\begin{cases} \frac{dq_n}{dt} & = p_n \\ \frac{dp_n}{dt} & = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \end{cases} \qquad n \in \mathbb{Z}$$

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- Infinitely many conserved quantities $\Rightarrow$ Generalized Gibbs Ensembles (GGE) and Generalized hydrodynamics (GHD)
- Random matrix representation of GGE and GHD for Toda (Spohn, 2020,2021,...)
- White-noise is invariant for KdV equation (Killip-Murphy-Visan, Invent. math. (2020)) . Well-posedness of the dynamics on $\mathbb{R}$ was one of obstacles. Ergodicity is still open.

Recent developments on BBS from statistical physics/probabilistic points of view

- Construction of bi-infinite dynamics（Existence and uniqueness of the dynamics）[Ferrari-Nguyen-Rolla-Wang (2021), Croydon-Kato-S-Tsujimoto (2023), Croydon-S-Tsujimoto (2022)]
- Construction of invariant measures, (periodic) Generalized Gibbs Ensembles : [Ferrari-Nguyen-Rolla-Wang (2021), Croydon-Kato-S-Tsujimoto (2023), Ferrari-Gabrielli (2020), Croydon-S (2019), Suda (2024+)]
- Derivations of macroscopic dynamics, in particular give a rigorous mathematical foundation of Generalized Hydrodynamics (GHD) : [Ferrari-Nguyen-Rolla-Wang (2021), Croydon-S (2021), Olla-S-Suda (2024+)]

1. Brief background on generalized hydrodynamic limits

Hydrodynamic limits

Consider αN particles independently performing continuous time asymmetric simple random walk on the torus $\mathbb{Z}/N\mathbb{Z}$, with drift $v \neq 0$. Write X_t^m for the position of the m-th particle at time t, and set the scaled empirical measure

$$\pi_t^N(du) := \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \sum_{m=1}^{\alpha N} \mathbf{1}_{\{X_{Nt}^m = x\}} \delta_{x/N}(du).$$

Suppose $\pi_0^N(du) \to \rho(u,0)du$ for some suitably smooth ρ, then

$$\pi_t^N(du) \to \rho(t,u)du,$$

where $\rho(t,u)$ $(= \rho(u - vt, 0))$ satisfies the partial differential equation (hydrodynamic equation):

$$\partial_t \rho = -\partial_u (v\rho).$$

E.g. [Kipnis-Landim (1999)]. Many similar results for interacting particle systems, e.g. zero range process, exclusion process.

Generalized Hydrodynamics

Extension of the theory of hydrodynamics to integrable systems with infinitely many conserved quantities, which was originally introduced for quantum integrable systems, see [Doyon (2020)] for a recent survey.

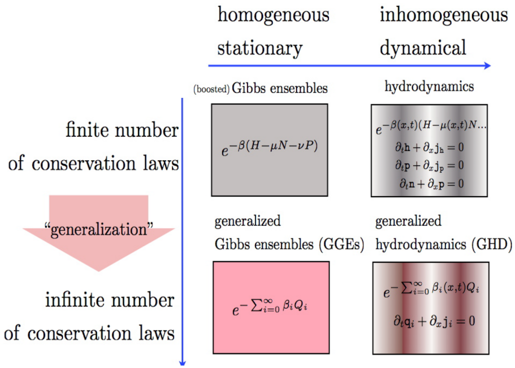


Figure : [Doyon (2020)]

GHD equations

GHD equations for $\rho = (\rho_p)_p$ where ρ_p is the macroscopic density of quasi particles of type p :

$$\begin{cases} \partial_t \rho_p = -\partial_u \left(v_p^{\mathrm{eff}}(\rho) \rho_p \right), \\ v_p^{\mathrm{eff}}(\rho) = v_p + \int \kappa(p,p') \rho_{p'} \left(v_{p'}^{\mathrm{eff}}(\rho) - v_p^{\mathrm{eff}}(\rho) \right) dr, \end{cases}$$

where:

- v_p is the speed of an isolated quasi particle of type p;
- $\kappa(p,p')$: phase shift (scattering, two-body shift)

GHD equations are expected to describe macroscopic dynamics for various types of integrable systems, including classical and quantum gases, chains, field theory models and cell automatons.

GHD equations (kinetic equation for a soliton gas) for KdV equation

Let $f_s = f_s(u,t)$ be the density at space-time point (u,t) of solitons with respect to their ‘spectral parameter’ s. These solve:

$$\begin{cases} \partial_t f_s = -\partial_u \left(v_s^{\mathrm{eff}}(f) f_s \right), \\ v_s^{\mathrm{eff}}(f) = v_s - \frac{1}{s} \int_0^\infty \kappa(s,r) f_r \left(v_r^{\mathrm{eff}}(f) - v_s^{\mathrm{eff}}(f) \right) dr, \end{cases}$$

where:

- $v_s = 4s^2$ is the speed of an isolated s-soliton;
- $\kappa(s,r) = \log |(s+r)/(s-r)|$ is an interaction kernel.

See [Zakharov, 1971], [El, 2003]. These can be seen as the ‘GHD equations’ for the soliton gas of KdV equation.

GHD equations for Toda equation

$$\begin{cases} \partial_t \rho_p = -\partial_u \left(v_p^{\text{eff}}(\rho) \rho_p \right), \\ v_p^{\text{eff}}(\rho) = v_p + \int \kappa(p, p') \rho_{p'} \left(v_{p'}^{\text{eff}}(\rho) - v_p^{\text{eff}}(\rho) \right) dr, \end{cases}$$

where:

- $v_p = p$ is the speed of an isolated s-soliton;
- $\kappa(p, p') = 2 \log |p - p'|$ is an interaction kernel.

See [Spohn], [Doyon]. These can be seen as the 'GHD equations' for the gas of quasi-particles of Toda equation.

'Cold-gas' reduction [El-Kamchatnov-Pavlov-Zykov, 2010]

Consider $f_s = \sum_{i=1}^{I} f_i \delta_{s_i}(s)$, where $0 < s_1 < s_2 < \cdots < s_I$.
Resulting system of hydrodynamic conservation laws:

$$\begin{cases} \partial_t \rho_i = -\partial_u \left(v_i^{\mathrm{eff}}(\rho) \rho_i \right), \\ v_i^{\mathrm{eff}}(\rho) = v_{s_i} - \sum_{j=1}^{I} \kappa_{ij} \rho_j \left(v_j^{\mathrm{eff}}(\rho) - v_i^{\mathrm{eff}}(\rho) \right), \end{cases}$$

where $\rho_i = s_i f_i$ and $\kappa_{ij} = \frac{1}{s_i s_j} \kappa(s_i, s_j)$.

We derive the exactly same form of equations from the box-ball system with $v_{s_i} = i$ and $\kappa_{ij} = 2(i \wedge j)$ as a space-time scaling limit.

Remark

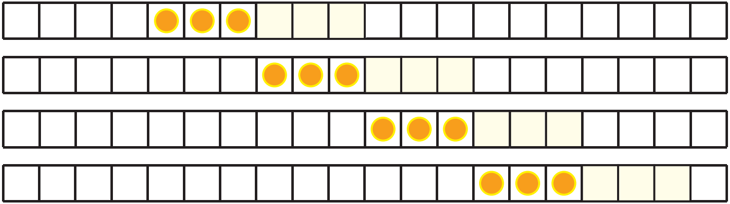
For hard rods, the corresponding result was obtained for homogeneous rods in 80's by Boldrighini, Dobrushin and Sukhov and for inhomogeneous rods recently by Ferrari, Franceschini, Grevino and Spohn (2023).

2. Solitonic behavior of the BBS

Solitons in the BBS

The BBS is an 'ultra-discretization' (i.e. zero temperature limit of a discretization) of the KdV equation, and also exhibits solitonic behavior:

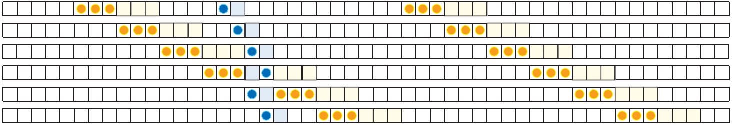
- $(1,0),(1,1,0,0),(1,1,1,0,0,0),\ldots$ are 'solitons'
- Call $(1,1,\ldots,1,0,0,\ldots,0)$ a length k soliton if it contains k copies of 1
- length k soliton moves with speed k



$v_k = k$: the speed of an isolated k-soliton

Soliton interaction in the BBS

Example interaction between two solitons:



When a size k soliton overtakes a size ℓ soliton ($k > \ell$), the larger soliton receives a push forward of $2\ell = 2\min\{k, \ell\}$, and the smaller a push back by the same amount.

$\kappa(k, \ell) := 2\min\{k, \ell\}$: the phase shift between k-soliton and ℓ-soliton

* Solitons will eventually line up in order of right to largest, and then each will simply move at its own speed.

* There are infinite number of conserved quantities : $\sharp$ of k-solitons for each $k \in \mathbb{N}$

Identifying solitons (Takahashi-Satsuma algorithm)

run := consecutive 0s or 1s

Let k be the length of the left-most run that is followed by a run of at least the same length. Group the elements of this run with the first k elements of the subsequent run. The $2k$ grouped elements are identified as a size k soliton. Remove the identified soliton, and repeat.

1 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 0 0

1 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 0 0

1 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 0 0

1 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 0 0

1 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 0 0

Linearization of the dynamics by the slot decomposition [Ferrari-Nguyen-Rolla-Wang, 2023]

Slot number (count how many 1's (or 0's)) from the left in the same soliton.

$$1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0$$

$$1\ 2\ 1\ 2$$

Slot configuration:

$$1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0$$

$$1\ 2\ 1\ 2\ 1\ 2\ 3\ 1\ 1\ 4\ 1\ 2\ 3\ 1\ 2\ 1\ 2\ 4\ \infty$$

Site x is a k-slot if the above number at x is bigger than k.
The effective distance between two size k-solitons $= \sharp\ k$ slots between them.

Slot decomposition [Ferrari-Nguyen-Rolla-Wang, 2023]

$\zeta_k(m) :=$ The number of k solitons between the m-th k-slot and $m+1$-th k-slot $=$ The number of k solitons at "the effective position m"

$\zeta = (\zeta_k)_{k \in \mathbb{N}}$ where $\zeta_k \in \mathbb{Z}_+^{\mathbb{Z}}$: the *slot decomposition* of η.

Importantly:

- This decomposition has an inverse (under some condition) ; $\eta \leftrightarrow \zeta$
- The dynamics of ζ is linearized : $\zeta_k(t, m) = \zeta_k(m - kt)$ for any time t

3. Invariant measures and Generalized Gibbs Ensembles for the BBS

Invariant measures of BBS via Pitman's transform description [Croydon-Kato-S-Tsujimoto, 2023]

BBS has a nice Pitman's transform description, which defines the dynamics for bi-infinite ball configurations with asymptotic ball density less than $\frac{1}{2}$

- I.i.d. Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with ball density $0 \leq p < \frac{1}{2}$ is $\mathcal{T}$-invariant.
- The distribution given by a two-sided stationary Markov chain on $\{0,1\}$ with $p_0 + p_1 < 1$ where $p_i = P(\eta_1 = 1 | \eta_0 = i)$ is $\mathcal{T}$-invariant.
- For each $K \in \mathbb{N}$, the distribution given by conditioning the i.i.d. Bernoulli product measure with ball density $p \in (0,1)$ on the event that there is no soliton greater than K is $\mathcal{T}$-invariant.

Formal Generalized Gibbs Ensembles for BBS

Formally, Generalized Gibbs Ensembles for BBS should be

$$P(\eta) = \frac{1}{Z_\beta} \exp\left(- \sum_{k=0}^{\infty} \beta_k f_k(\eta) \right)$$

$f_0(\eta)$: Number of balls in η
$f_k(\eta)$: Number of solitons with size $\geq k$ in η
In particular,

$$f_0(\eta) = \sum_{x \in \mathbb{Z}} \eta_x, \quad f_1(\eta) = \sum_{x \in \mathbb{Z}} \mathbf{1}_{(\eta_x, \eta_{x+1}) = (1,0)}$$

- Bernoulli product measure : $\beta_0 = \log\left(\frac{1-p}{p}\right), \beta_k = 0 \ (k \geq 1)$
- Stationary Markov chain :
  $\beta_0 = \log\left(\frac{1-p_0}{p_1}\right), \beta_1 = \log\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right), \beta_k = 0 \ (k \geq 2)$
- Bernoulli product measure conditioned to solitons of size $\leq K$:
  $\beta_0 = \log\left(\frac{1-p}{p}\right), \beta_k = 0 \ (1 \leq k \leq K), \beta_k = \infty \ (k > K)$

Periodic Generalized Gibbs Ensembles for BBS

P^N be the probability measure on the configurations with period N as

$$P^N(\eta) = \frac{1}{Z_\beta^N} \exp\left(-\sum_{k=0}^{\infty} \beta_k f_k^N(\eta) \right) \mathbf{1}_{\{\sum_{x=1}^N \eta_x < \frac{N}{2}\}}.$$

Theorem

For any β, P^N is invariant under $\mathcal{T}$.

For three examples in the last slide, P_N converges to the corresponding probability measure as $N \to \infty$. [Croydon-S, 2019]

Remark

Except the three special cases, the convergence of P_N to the corresponding probability measure is not yet proved.

Invariant measures via solitons [Ferrari-Nguyen-Rolla-Wang, 2023]

Let $\zeta = (\zeta_k)_{k \geq 1}$ be independent random elements of $\mathbb{Z}_+^{\mathbb{Z}}$ with shift-invariant distributions satisfying

$$\sum_k k\mathbf{E}(\zeta_k(0)) < \infty, \qquad \mathbf{P}\left(\sum_{k,m} \zeta_k(m) > 0\right) = 1.$$

Then there exists a unique shift-invariant probability measure on η such that η has soliton decomposition ζ. This measure is $\mathcal{T}$-invariant. Moreover, if $(\zeta_k(m))_{m \in \mathbb{Z}}$ is i.i.d. for each k, then this measure is shift-ergodic.

Note : Given a sequence $(\rho_k)_{k \in \mathbb{N}}$ specifying the density of k-solitons, we can construct an infinite number of mutually singular shift-invariant and $\mathcal{T}$-invariant measures, all having the same specified soliton densities.

Soliton decomposition in i.i.d. case and Markov chain case [Ferrari-Gabrielli, 2020]

- For η i.i.d. Bernoulli with density $p < 1/2$, the elements of $(\zeta_k(m))_{k,m}$ are independent. Moreover, $\zeta_k(m)$ is geometric, with parameter $1 - q_k$, where $q_1 := p(1-p)$ and

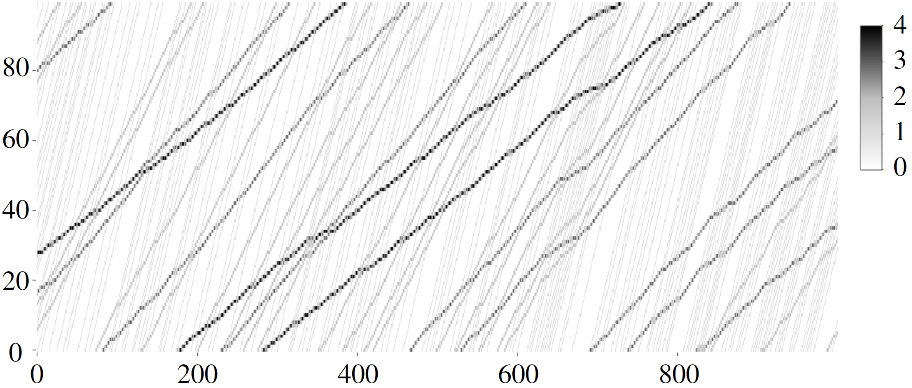
$$q_k := \frac{(p(1-p))^k}{\prod_{\ell=1}^{k-1}(1-q_\ell)^{2(k-\ell)}}, \qquad k \geq 2.$$

- For η : stationary Markov chain, the elements of $(\zeta_k(m))_{k,m}$ are independent. Moreover, $\zeta_k(m)$ is geometric, with some parameters given by p_0, p_1.

Note : Given a sequence $(\rho_k)_{k\in\mathbb{N}}$ specifying the density of k-solitons (satisfying some condition), there is a unique shift-invariant and $\mathcal{T}$-invariant measure under which $(\zeta_k(m))_{k,m}$ are independent and $\zeta_k(m)$ is geometric. These measures are shift-ergodic. They should be "Generalized Gibbs Ensembles" for BBS.

4. Euler-scale dynamics of solitons

Soliton speeds with random initial condition : simulation



The figure shows the numerical simulation result for initial configuration $\eta = (\eta_x)_{x \in \mathbb{Z}}$ a realization of a sequence of i.i.d. Bernoulli(0.2) random variables. By Croydon.

Effective speeds [Ferrari-Nguyen-Rolla-Wang, 2023]

Let the initial configuration η follow a $\mathcal{T}$-invariant and shift-ergodic measure with $(\rho_k)_{k\geq}$: the densities of solitons of different sizes.

$X_k(t)$ be the position of a tagged k-soliton at time t, then

$$\lim_{t\to\infty} \frac{X_k(t)}{t} = v_k^{\mathrm{eff}}(\rho) \quad \text{almost surely}$$

where the effective speeds $(v_k^{\mathrm{eff}}(\rho))_{k\geq 1}$ satisfy:

$$v_k^{\mathrm{eff}}(\rho) = v_k - \sum_{\ell=1}^{\infty} \kappa_{k,\ell}\rho_\ell(v_\ell^{\mathrm{eff}}(\rho) - v_k^{\mathrm{eff}}(\rho)),$$

where $v_k := k$ and $\kappa_{k,\ell} = 2(k \wedge \ell)$

Note: The above equation for $(v_k^{\mathrm{eff}}(\rho))_{k\geq 1}$ may have multiple solutions. In the case when soliton sizes are bounded by K, we have

$$M(\rho)v^{\mathrm{eff}}(\rho) = v$$

for a suitable $K \times K$ matrix $M(\rho)$ which is invertible, and so $v^{\mathrm{eff}}(\rho) = M(\rho)^{-1}v$ is the unique solution.

Recall : Linearization [TAKAHASHI/SATSUMA], [FERRARI/NGUYEN/ROLLA/WANG]

Write $S_i(x) = \#\{i\text{-slots up to spatial position } x\}$. The number of i solitons in the mth i-slot is then:

$$\zeta_i(m) = \sum_{x=1}^{\infty} \sigma_i(x) \mathbf{1}_{\{S_i(x)=m\}}.$$

This is the *slot decomposition* of η, where
$\sigma_i(x) = 1_{x \text{ is the left-most site of a } i\text{-soliton}}.$
The cumulative number of i solitons in terms of slots is:

$$\bar{\psi}_i(z) := \sum_{m=1}^{z} \zeta_i(m).$$

Importantly:

- this decomposition has an inverse;
- it holds that the configuration $T^t\eta$ corresponds to

$$\bar{\psi}_i(z - it), \qquad i = 1, 2, \ldots .$$

SCATTERING MAP (INTUITION!)

Connection between spatial and slot pictures, i.e. description of the scattering map $(\psi_i) \mapsto (\bar{\psi}_i)$? We have:

$$\begin{aligned}
S_i(u) &= \#\{i\text{-slots up to spatial position } u\} \\
&\approx R(u) + \sum_{j>i} 2(j-i)\psi_j(u) \\
&\approx u - \sum_{j=1}^{\infty} 2j\psi_j(u) + \sum_{j>i} 2(j-i)\psi_j(u) \\
&= u - \sum_{j=1}^{\infty} 2(i \wedge j)\psi_j(u) \\
&=: \phi_i(u).
\end{aligned}$$

So

$$\bar{\psi}_i \approx \psi_i \circ \phi_i^{-1}.$$

The continuous scattering map

The state will be encoded by $(\psi_i)_{i=1}^I \in \mathcal{D}$, where ψ_i represents the *integrated density of size i solitons*.

Effective distance accumulated by size i solitons over the interval $[0, u]$:

$$\phi_i(u) := u - \sum_{j=1}^{I} 2(i \wedge j)\psi_j(u), \qquad i = 1, 2, \ldots, I.$$

Above:

$$\mathcal{D} := \left\{ (\psi_i)_{i=1}^I \in \mathcal{C}^I : \phi_I \in \mathcal{C}^\uparrow, \sum_{i=1}^{I} i \sup_{u_1, u_2} \frac{\psi_i(u_1) - \psi_i(u_2)}{\phi_i(u_1) - \phi_i(u_2)} < \frac{1}{2} \right\}.$$

where: $\mathcal{C} = \{$continuous, non-decreasing, started from 0$\}$,
$\mathcal{C}^\uparrow = \{$continuous, strictly increasing, started from 0, divergent$\}$.

CONTINUOUS DYNAMICS

Write Υ for the 'scattering' map that takes $(\psi_i)_{i=1}^I \in \mathcal{D}$ to $(\bar{\psi}_i)_{i=1}^I \in \mathcal{C}^I$, where

$$\bar{\psi}_i := \psi_i \circ \phi_i^{-1}$$

is the *integrated density of solitons on their effective scale*.

Proposition. The map $\Upsilon : \mathcal{D} \to \mathcal{C}^I$ is a bijection, with an explicit inverse.
Define $\theta_t : \mathcal{C}^I \to \mathcal{C}^I$ by

$$(\theta_t \circ \bar{\psi})_i(z) = \bar{\psi}_i((z - it) \vee 0).$$

DYNAMICS. If $\psi_i(\cdot, 0) = \psi_i^0$, $i = 1, 2, \ldots, I$, then

$$\psi_i(u, t) := \left(\Upsilon^{-1} \circ \theta_t \circ \Upsilon \circ \psi^0\right)_i (u).$$

P.D.E.

Suppose $(\psi_i(u,t))$ has initial condition

$$\psi_i(u,0) = \psi_i^0(u) := \int_0^u \rho_i^0(x)dx,$$

and satisfies

$$\psi_i(u,t) := \left(\Upsilon^{-1} \circ \theta_t \circ \Upsilon \circ \psi^0\right)_i (u).$$

Then (under various regularity conditions) $\rho_i(u,t) := \partial_u \psi_i(u,t)$ is the unique classical solution of the partial differential equation

$$\begin{cases} \partial_t \rho_i = -\partial_u \left(v_i^{\mathrm{eff}}(\rho)\rho_i \right), \\ \rho_i(\cdot, 0) = \rho_i^0(\cdot), \end{cases} \qquad i = 1, 2, \ldots, I,$$

Proof. Follows from the obvious fact that $\partial_t(\bar{\psi}_i) = -v_i \partial_z(\bar{\psi}_i)$.

Generalized hydrodynamic limit [Croydon-S, 2020]: assumptions

Fix $K \in \mathbb{N}$. Let $\rho^0 = (\rho_k^0)_{k=1}^K$ satisfy some total density condition and the regularity condition. For each $N \in \mathbb{N}$, consider the BBS starting from a (random) configuration $\eta^N \in \Omega_K$ where

$$\Omega_K := \{\eta = (\eta_x)_x \in \{0,1\}^{\mathbb{Z}} \mid \eta_x = 0 \ (\forall x \leq 0), \text{no soliton with size} > K\}.$$

$$\sigma_k^N(x,t) := \mathbf{1}_{\{\exists \text{ a soliton of size } k \text{ in } T^t \eta^N \text{ starting at spatial location } x\}}.$$

and set

$$\pi_k^{N,t}(du) := \frac{1}{N} \sum_{x \in \mathbb{N}} \sigma_k^N(x, \lfloor Nt \rfloor)\, \delta_{x/N}(du), \qquad u, t \in \mathbb{R}_+.$$

Suppose that, for every $(F_k)_{k=1}^K \in C_0(\mathbb{R}_+, \mathbb{R})^I$,

$$\lim_{N\to\infty} \left| \int_{\mathbb{R}_+} F_k(u) \pi_k^{N,0}(du) - \int_{\mathbb{R}_+} F_k(u) \rho_k^0(u) du \right| = 0 \text{ in prob.}$$

Generalized hydrodynamic limit [Croydon-S, 2020]: conclusion

It then holds that, for every $t \in (0,\infty)$ and $(F_k)_{k=1}^K \in C_0(\mathbb{R}_+,\mathbb{R})^K$,

$$\lim_{N\to\infty} \left| \int_{\mathbb{R}_+} F_k(u)\pi_k^{N,t}(du) - \int_{\mathbb{R}_+} F_k(u)\rho_k(u,t)du \right| = 0 \text{ in prob.},$$

where $(\rho_k(u,t))_{u,t\in\mathbb{R}_+,k=1,2,\dots,K}$ is the unique classical solution of the partial differential equation

$$\begin{cases} \partial_t \rho_k = -\partial_u \left(v_k^{\mathrm{eff}}(\rho)\rho_k \right), \\ \rho_k(\cdot,0) = \rho_k^0(\cdot), \end{cases} \qquad k = 1,2,\dots,K,$$

amongst the class of functions $\rho \in C^1(\mathbb{R}_+^2,\mathbb{R}_+)^K$ satisfying some total density condition for all $t \geq 0$.

Remarks/Open problems

1. Ongoing work to extend to $K = \infty$. (Mainly technical.)

2. For two-sided case, need to handle flow across the origin 0. This will also allow one to connect with GHD of [Kuniba-Misguich-Pasquier, 2020].

3. For the finite capacity BBS, we can prove the same result by using our new results which connect KKR bijection and the slot decomposition (Mucciconi-S-Sasamoto-Suda, 2024)

GHD equation starting from the domain-wall initial condition [Kuniba-Misguich-Pasquier, 2020]

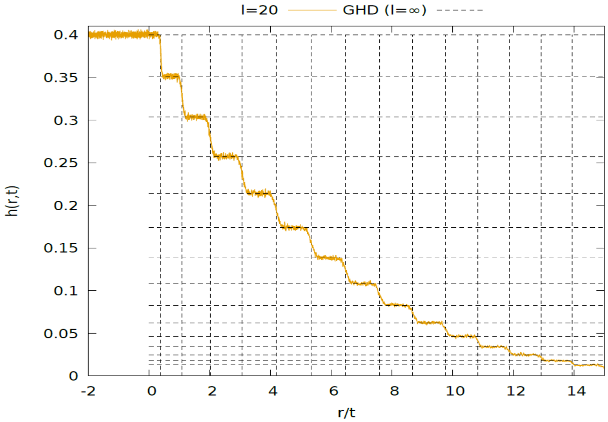


Figure 7. Ball density for the capacity $l = 20$ at fixed $t = 500$. Initial ball densities: $p_L = 0.4$ and $p_R = 0$. The dotted lines correspond to the horizontal and vertical positions of the plateaux predicted by the GHD approach in the limit $l = \infty$, see (5.43) and (5.44).

5. Fluctuation of tagged solitons

On going work with S.Olla and H. Suda

Assume that the initial distribution ν of η is a Bernoulli product measure or two-sided Markov distribution with ball density less than $\frac{1}{2}$.

$X_k(t)$ be the position of a tagged k-soliton at time t, then

- the process $t \to X_k(t) - v_k^{\mathrm{eff}}(\nu)t$ converges to a Brownian motion under the diffusive space-time scale. The diffusion coefficient is computable, but complicated.
- For two different k-solitons, $X_k(t)$ and $X'_k(t)$, if $|X_k(0) - X'_k(0)| = O(N)$, then their fluctuations converges to the same Brownian motion under the diffusive space-time scale.
- $\frac{X_k(t)}{t}$ satisfies the large deviation principle with a certain good rate function.

6. Key ideas

Macroscopic scattering map

- Slot decomposition = Microscopic scattering and inverse scattering maps
- We construct macroscopic scattering and inverse scattering maps $(\rho_k(u))_{k\in\mathbb{N}} \leftrightarrow (\bar{\rho}_k(u))_{k\in\mathbb{N}}$ explicitly
- $\bar{\rho}_k(u,t) = \bar{\rho}_k(u - v_k t)$.
- The relation between these scattering/inverse scattering maps and the GDH equation is universal!

Relation between different linearization methods

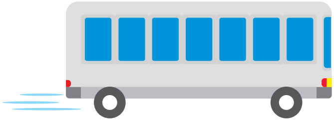
Known linearization methods

- KKR-bijection : complicated, BBS as a quantum integrable system
- 10-elimination : intuitively simple, but mathematically not

New linearization method

- Slot decomposition : Depends heavily on Takahashi-Satsuma algorithm, BBS as a classical integrable system, useful in statistical physics and probabilistic approaches

To connect them, we introduced the seat number configuration (Mucciconi-S-Sasamoto-Suda, 2024).



1 1 0 0 1 1 1 0 1 . . .

Seat number and slot configuration

Recall:

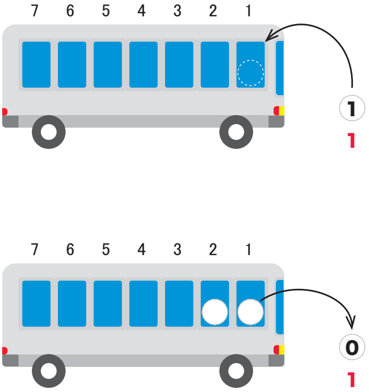
$$1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0$$

$$1\ 2\ 1\ 2\ 1\ 2\ 3\ 1\ 1\ 4\ 1\ 2\ 3\ 1\ 2\ 1\ 2\ 4\ \infty$$

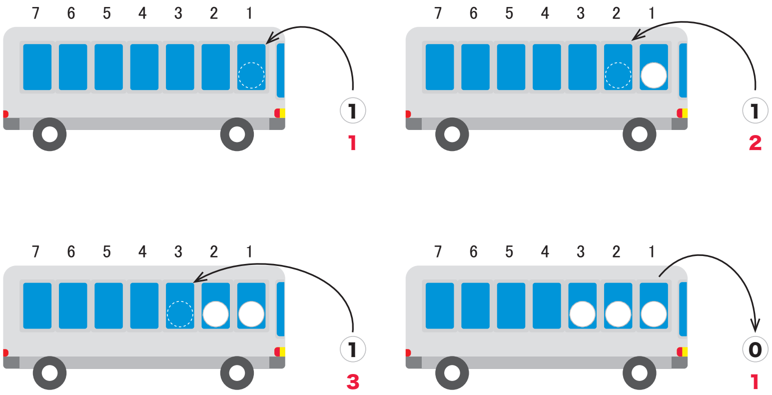
How to obtain these numbers without Takahashi-Satsuma algorithm?

- Consider a carrier with seat numbers
- The ball that picked up always sits in the seat with the smallest seat number of the vacant seats.
- When put down the ball from the carrier, put down the one at the smallest seat number.
- Record the number of the seat number where the occupation variable has changed.

Seat number algorithm



Seat number algorithm



10-elimination

$$1\ \cancel{1\ 0}\ 0\ 1\ 1\ \cancel{1\ 0}\ 1\ \cancel{1\ 0}\ 0\ 0\ 1\ \cancel{1\ 0}\ 0\ 0\ 0$$

Record all the places where $(1, 0)$ are in this order and erase them.

10-elimination

1 ~~1 0~~ 0 1 1 ~~1 0~~ 1 ~~1 0~~ 0 0 1 ~~1 0~~ 0 0 0

Record all the places where $(1, 0)$ are in this order and erase them.

Original configuration: 1 size 1-soliton, 2 size 2-soliton, 1 size 4-soliton

1 ~~1 0~~ 0 1 1 ~~1 0~~ 1 ~~1 0~~ 0 0 1 ~~1 0~~ 0 0 0

1 0 1 1 1 0 0 1 0 0 0

1 0 1 1 1 0 0 1 0 0 0

New configuration: 2 size 1-soliton, 1 size 3-soliton
Repeat this procedure and collect all the information.

10-elimination and 1-seat elimination [Suda, 2024+]

~~1~~ 1 ~~0~~ 0 ~~1~~ 1 1 ~~0~~ ~~1~~ 1 ~~0~~ 0 0 ~~1~~ 1 ~~0~~ 0 0 0

~~1~~ 2 ~~1~~ 2 ~~1~~ 2 3 ~~1~~ ~~1~~ 4 ~~1~~ 2 3 ~~1~~ 2 ~~1~~ 2 4 ∞

Erase all the sites with the seat number 1.

10-elimination and 1-seat elimination [Suda, 2024+]

~~1~~ 1 ~~0~~ 0 ~~1~~ 1 1 ~~0~~ ~~1~~ 1 ~~0~~ 0 0 ~~1~~ 1 ~~0~~ 0 0 0

~~1~~ 2 ~~1~~ 2 ~~1~~ 2 3 ~~1~~ ~~1~~ 4 ~~1~~ 2 3 ~~1~~ 2 ~~1~~ 2 4 ∞

Erase all the sites with the seat number 1.

Original configuration: 1 size 1-soliton, 2 size 2-soliton, 1 size 4-soliton

~~1~~ 1 ~~0~~ 0 ~~1~~ 1 1 ~~0~~ ~~1~~ 1 ~~0~~ 0 0 ~~1~~ 1 ~~0~~ 0 0 0

~~1~~ 2 ~~1~~ 2 ~~1~~ 2 3 ~~1~~ ~~1~~ 4 ~~1~~ 2 3 ~~1~~ 2 ~~1~~ 2 4 ∞

1 0 1 1 1 0 0 1 0 0 0

1 1 1 2 3 1 2 1 1 3 ∞

New configuration: 2 size 1-soliton, 1 size 3-soliton
The new seat number = the old seat number -1

Decomposition of fluctuation

- The recursive structure of 1-seat elimination and the i.i.d. property of the slot decomposition $\zeta_k(m)$ are key ingredients.
- Fluctuation of the position of a tagged k-soliton is decomposed as (i) the sum of the part about the interactions with solitons larger than size k and (ii) the part about the interactions with size ℓ-soliton for $1 \leq \ell \leq k-1$. These solitons can be considered as a random environment.
- The part about the interactions with solitons larger than size k is independent from the smaller solitons. But the scaling limit of this part is not trivial. For Bernoulli or Markov distribution case, we can use the existing result of the ergodicity for the current of balls in [Croydon-Kato-S-Tsujimoto].
- The part about the interactions with size ℓ-soliton can be reduced to the i.i.d. sum where the number of terms is random and depends on the larger solitons. It is not too difficult to prove that this part converges to a Brownian motion.