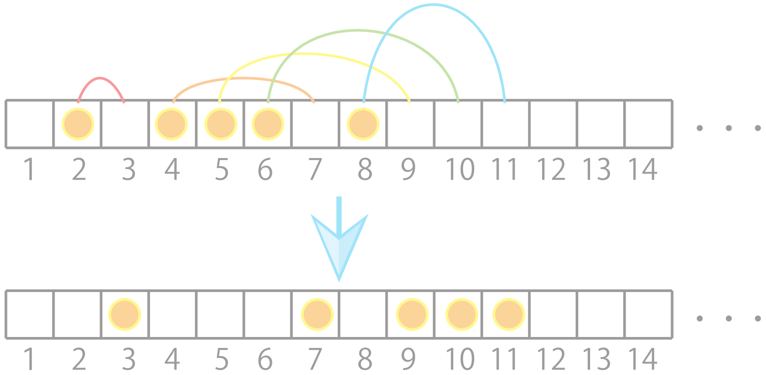
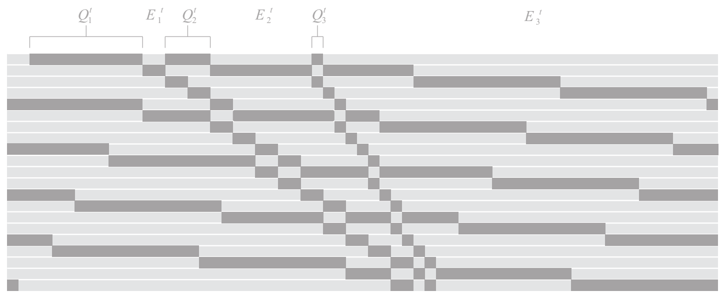
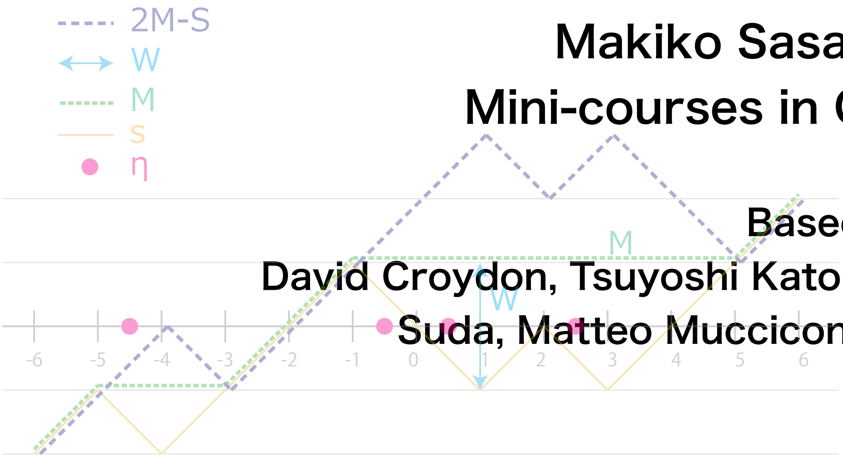




Introduction to probabilistic aspects of integrable systems



Makiko Sasada (University of Tokyo)

Mini-courses in GSSI trimester Day 1 @ GSSI

Based on joint work with
David Croydon, Tsuyoshi Kato, Satoshi Tsujimoto, Ryosuke Uozumi, Hayate Suda, Matteo Mucciconi, Tomohiro Sasamoto, Stefano Olla,…

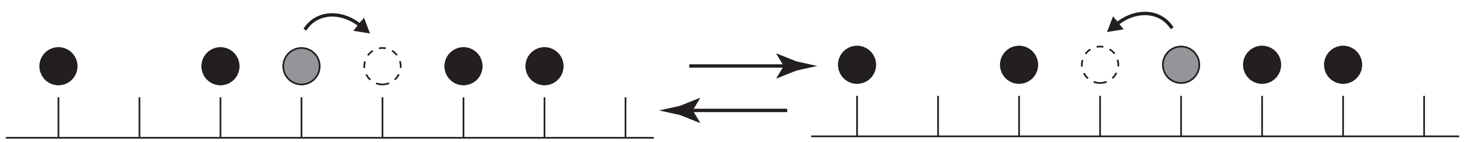
Interacting particle systems

Goal : Derive macroscopic evolution equations from microscopic dynamics of particles

★ Microscopic dynamics is typically a stochastic process

★ Macroscopic deterministic PDE is derived by a proper space-time scaling limit as a law of large numbers (Hydrodynamic limit)

★ The scaling limits of the microscopic fluctuations (central limit theorem) around the hydrodynamic limit are interesting research subjects : EW universality, KPZ universality,⋯



Main message of this short course

in space and time

deterministic

Discrete integrable systems are interesting!

★ They are a rich source of probabilistic problems!

★ They spotlight some classical subjects/topics in probability again! eg. Pitman(1974), Lukacs(1955), Crawford(1966)⋯

★ They are "mathematically tractable" models for testing Generalized Hydrodynamics (GHD) (2016-, Doyon, Spohn⋯).

★ They may have an exciting relationship with models in the integrable probability.

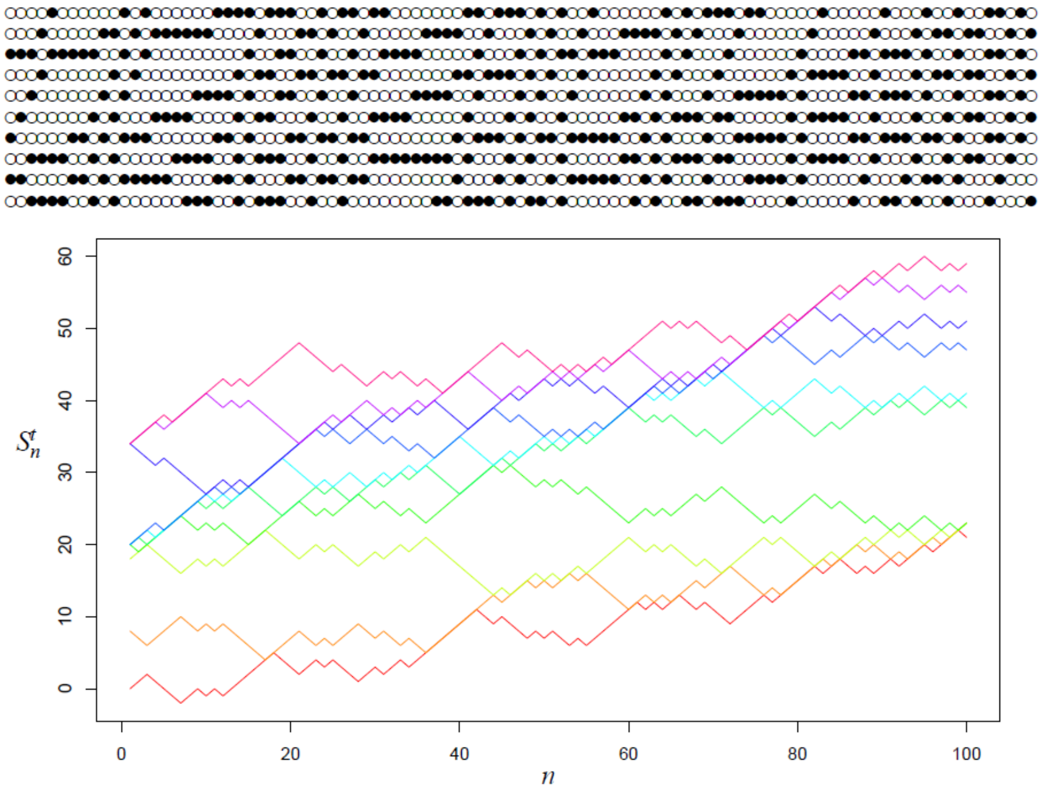
The simplest model of discrete integrable systems is

"Box-ball system"

Plan

1. Box-ball system
2. Other discrete integrable systems
3. General frameworks and theorems for invariant measures
4. Generalized Hydrodynamics for BBS

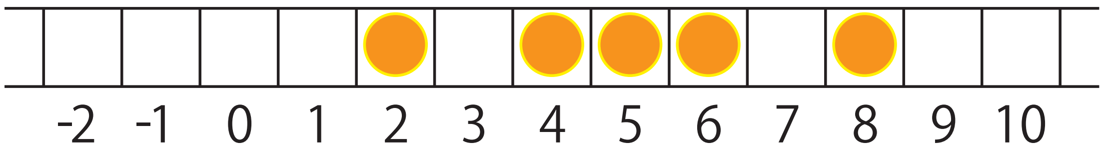
1. Box-ball system



Box-ball system (BBS)

Introduced in 1990 by Takahashi-Satsuma

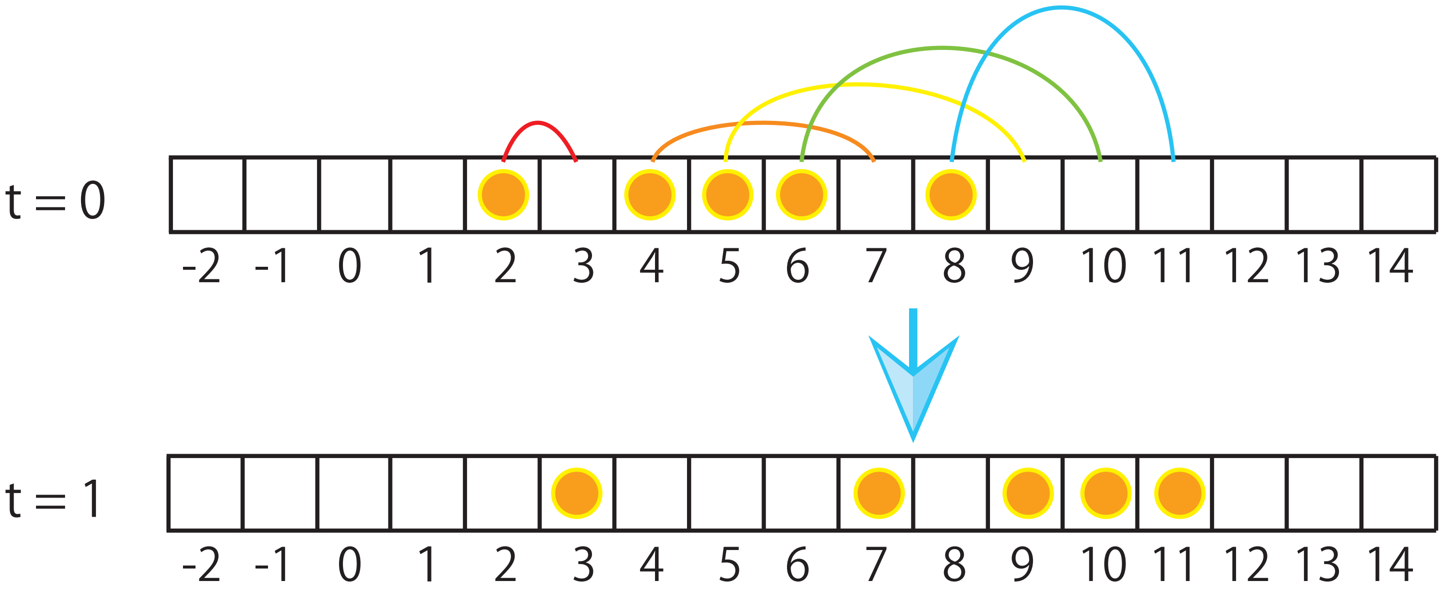
State space : $\Omega_{\mathbf{f}} := \left\{ \eta = (\eta_n)_n \in \{0,1\}^{\mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} \eta_n < \infty \right\}$



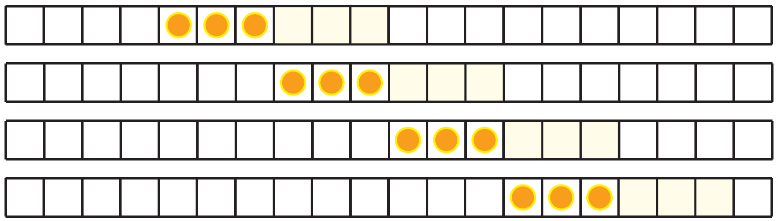
- **Finite number of balls**
- **Discrete time**
- **Deterministic dynamics**

Dynamics of BBS

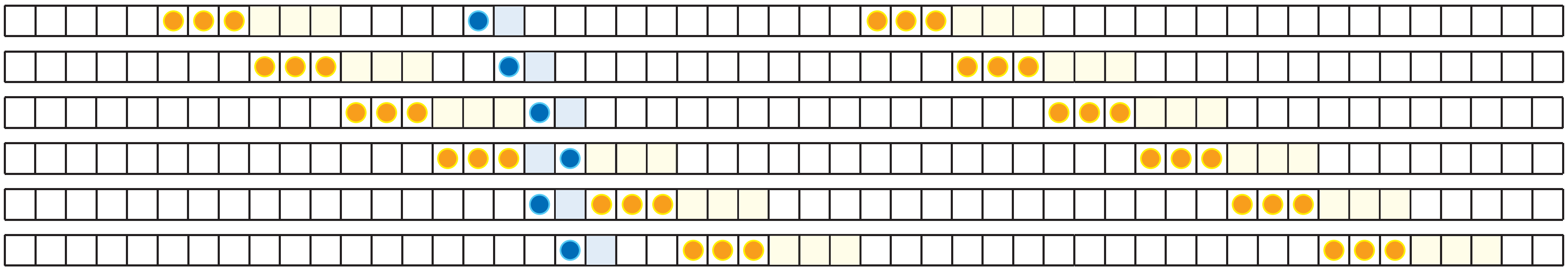
- Every ball moves exactly once in each time step.
- The leftmost ball moves first, the next leftmost ball moves next, and so on...
- Each ball moves to its nearest right vacant box.



Solitons of BBS



- (1,0), (1,1,0,0), (1,1,1,0,0,0) . . . are 'solitons'
- Call (1,1, . . . ,1, 0,0, . . . ,0) a size k soliton if it contains k copies of 1
- Size k soliton moves with speed k (speed in isolation)



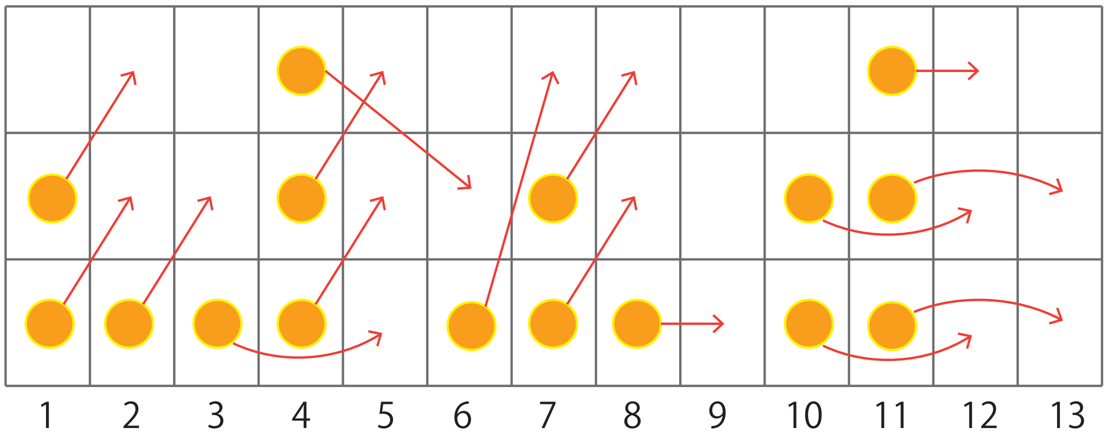
When size k soliton and size m soliton interact, bigger soliton is pushed forward by $2\min\{k,m\}$ sites and smaller soliton is pushed back by $2\min\{k,m\}$ sites. (phase shift)

Moreover, a three (or more)-soliton interaction is factorized into well separated two-soliton interactions. (Yang-Baxter equation)

Known properties of BBS

- The BBS has been deeply studied from algebraic points of view.
- Any configuration is decomposed into solitons.
- Initial value problem is solvable by the inverse scattering method.
- For all k, the number of size k solitons are conserved. Hence, the BBS has infinitely many conserved quantities.

- The BBS is reversible as a dynamical system. Well-defined for time $t \in \mathbb{Z}$.
- The BBS is the “ultra-discretization” of the discrete KdV equation.
- There are many integrable variants of the BBS.



“Equation of motion” of BBS

$\eta \in \Omega_{\mathbf{f}}$: ball configuration $\quad \Omega_{\mathbf{f}} := \left\{ \eta = (\eta_n)_n \in \{0,1\}^{\mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} \eta_n < \infty \right\}$

The BBS dynamics (one time-step) map : $T : \Omega_{\mathbf{f}} \to \Omega_{\mathbf{f}}$

$$\begin{cases} T\eta_n = 0 \quad \text{if} \quad \eta_m = 0 \ \forall m \le n \\ T\eta_n = \min \left\{ 1 - \eta_n \ , \ \sum_{m=-\infty}^{n-1} (\eta_m - T\eta_m) \right\} \end{cases}$$

Non-local. Infinite sum.

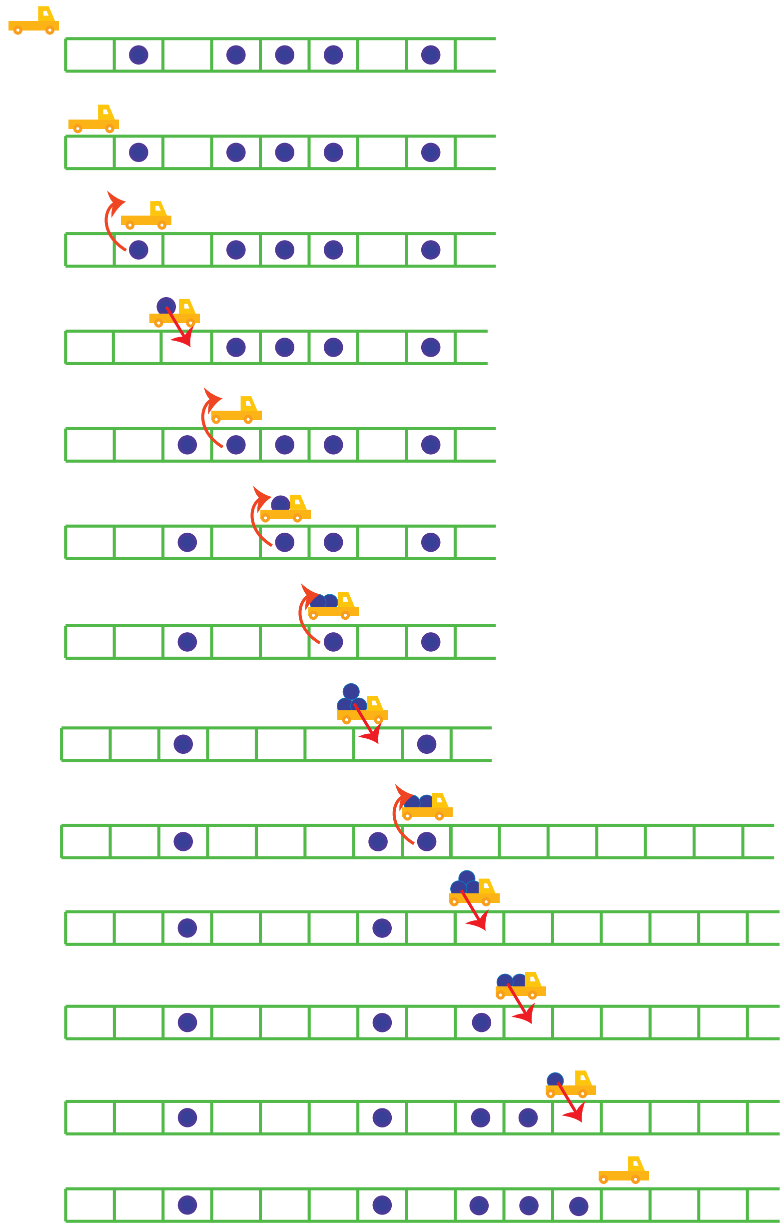
Probabilistic point of view : No invariant measure on Ω_f !!

Need to define the BBS with infinite balls (bi-infinite BBS).

Dynamics of BBS (Definition 2)

- A carrier goes through from left to right.
- The carrier picks up a ball if it finds a ball.
- The carrier puts down a ball if it comes to an empty box when it carries at least one ball.

Key description for probabilistic approach



Auxiliary variable W : carrier

W_n : The number of balls on the carrier as it passes location n, which is actually a current of balls

$$W_n = \sum_{m=-\infty}^{n} (\eta_m - T\eta_m)$$

Local conservation of the number of balls

$$\begin{cases} T\eta_n = \min\{1-\eta_n, W_{n-1}\} \\ W_n = W_{n-1} + \eta_n - T\eta_n = W_{n-1} + \eta_n - \min\{1-\eta_n, W_{n-1}\} \end{cases}$$

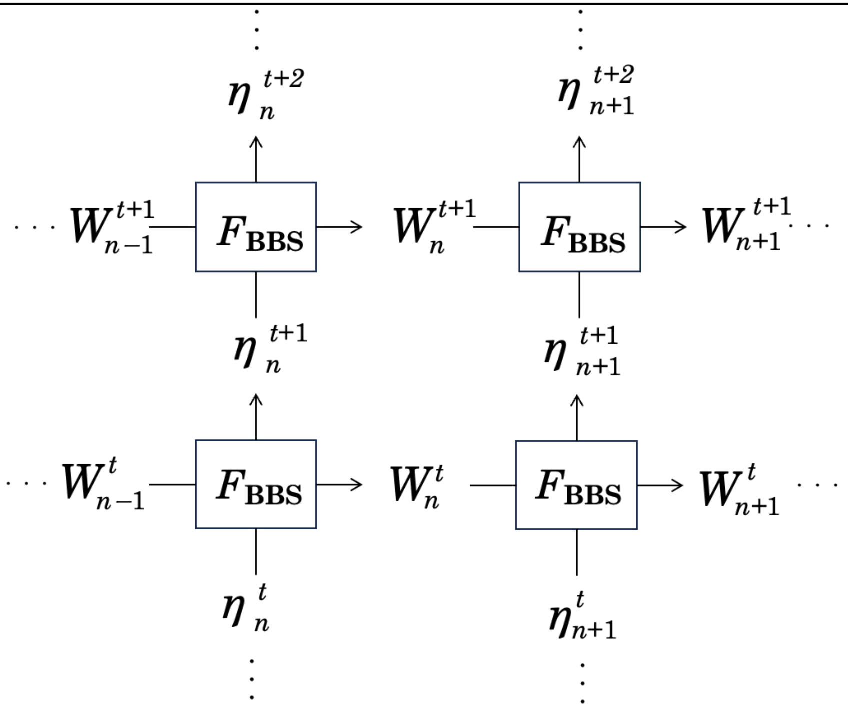
$$(T\eta_n, W_n) = F_{\mathrm{BBS}}\,(\eta_n, W_{n-1})$$

$$F_{\mathrm{BBS}}\,(a,b) = (\min\{1-a,b\}, a+b-\min\{1-a,b\}), \quad F_{\mathrm{BBS}} = F_{\mathrm{BBS}}^{-1}$$

2-dimensional lattice description of BBS

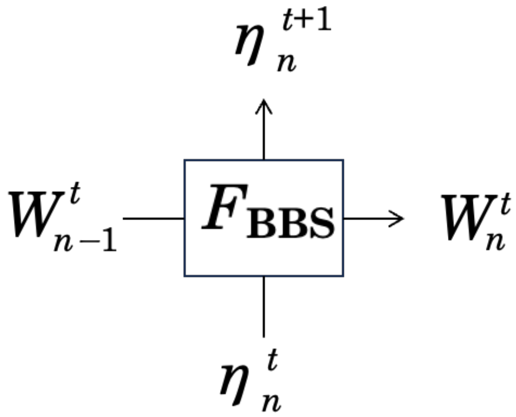
Initial value problem for the BBS : For a given $\eta \in \{0,1\}^{\mathbb{Z}}$, $(\eta_n^t, W_n^t)_{n,t\in\mathbb{Z}}$ is a solution of the initial value problem of the BBS if
$$\begin{cases} \eta_n^0 = \eta_n & \forall n \in \mathbb{Z} \\ (\eta_n^{t+1}, W_n^t) = F_{\mathrm{BBS}}(\eta_n^t, W_{n-1}^t) & \forall n, t \in \mathbb{Z} \end{cases}$$

Q : When a solution exists?
Is it unique?



Initial value problem for BBS

$$\begin{cases} \eta_n^0 = \eta_n & \forall n \in \mathbb{Z} \\ (\eta_n^{t+1}, W_n^t) = F_{\mathrm{BBS}}(\eta_n^t, W_{n-1}^t) & \forall n, t \in \mathbb{Z} \end{cases}$$



$$\Omega := \left\{ \eta \in \{0,1\}^{\mathbb{Z}} \mid \exists \lim_{n \to \pm\infty} \frac{\sum_{k=0}^{n} \eta_k}{n} < \frac{1}{2} \right\}.$$

Theorem (Croydon-S-Tsujimoto 2022, cf. Ferrari-Nguyen-Rolla-Wang, Croydon-Kato-S-Tsujimoto)

Suppose $\eta \in \Omega$, then there exists a unique solution, and $\eta^t \in \Omega$ for any $t \in \mathbb{Z}$.

The dynamics agrees with the finite BBS and also with the periodic BBS.

Key of the proof : Pitman's transform

Pitman's transform

<u>One-sided version</u> : $S : \mathbb{R}_{\geq 0} \to \mathbb{R},\ S_0 = 0$, continuous

$M_x := \max_{0 \leq y \leq x} S_y, \quad TS_x := 2M_x - S_x$: reflection w.r.t. the past maximum

Theorem (Pitman, 1975)

S : Brownian motion $\Rightarrow$ TS : 3-dimensional Bessel process

<u>Two-sided version</u> : $S : \mathbb{R} \to \mathbb{R},\ S_0 = 0$, continuous

$$M_x := \max_{y \leq x} S_y, \quad TS_x := 2M_x - S_x - 2M_0$$

Theorem (Harrison-Williams, 1987)

S : Brownian motion + positive drift $\Rightarrow$ $S \overset{(d)}{=} TS$

Exponential version of Pitman's transform

<u>One-sided version</u> : $S : \mathbb{R}_{\geq 0} \to \mathbb{R},\ S_0 = 0$, continuous

$$M_x := \log \int_0^x \exp(S_y)dy, \quad TS_x := 2M_x - S_x$$

Theorem (Matsumoto-Yor, 2000)

S : Brownian motion $\Rightarrow$ TS : Brownian motion in exponential potential

<u>Two-sided version</u> : $S : \mathbb{R} \to \mathbb{R}$, continuous

$$M_x := \log \int_{-\infty}^x \exp(S_y)dy, \quad TS_x := 2M_x - S_x - 2M_0$$

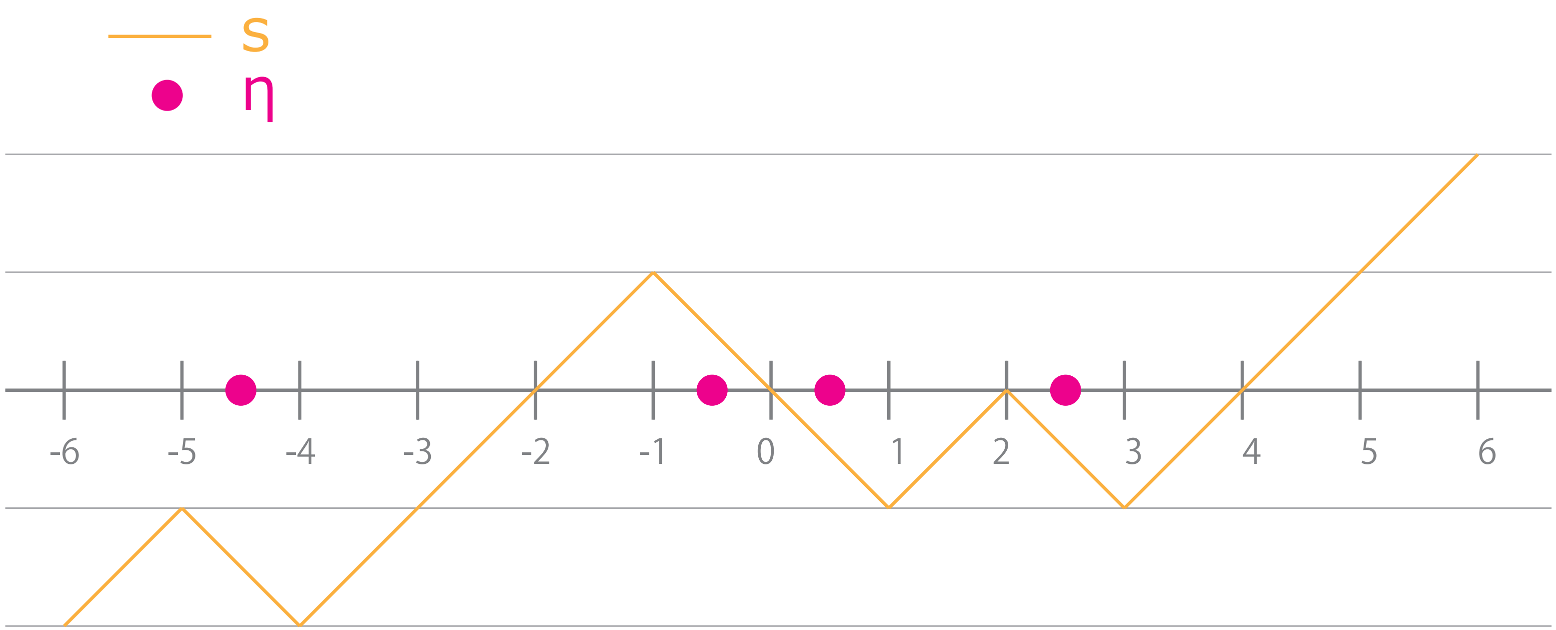
Theorem (O'Connell-Yor, 2001)

S : Brownian motion +positive drift $\Rightarrow$ $S \overset{(d)}{=} TS$

Path encoding of ball configurations of BBS

Path encoding : $S_0 = 0,\ S_n - S_{n-1} = 1 - 2\eta_n \in \{1, -1\}$

$S \leftrightarrow \eta$ **one to one**

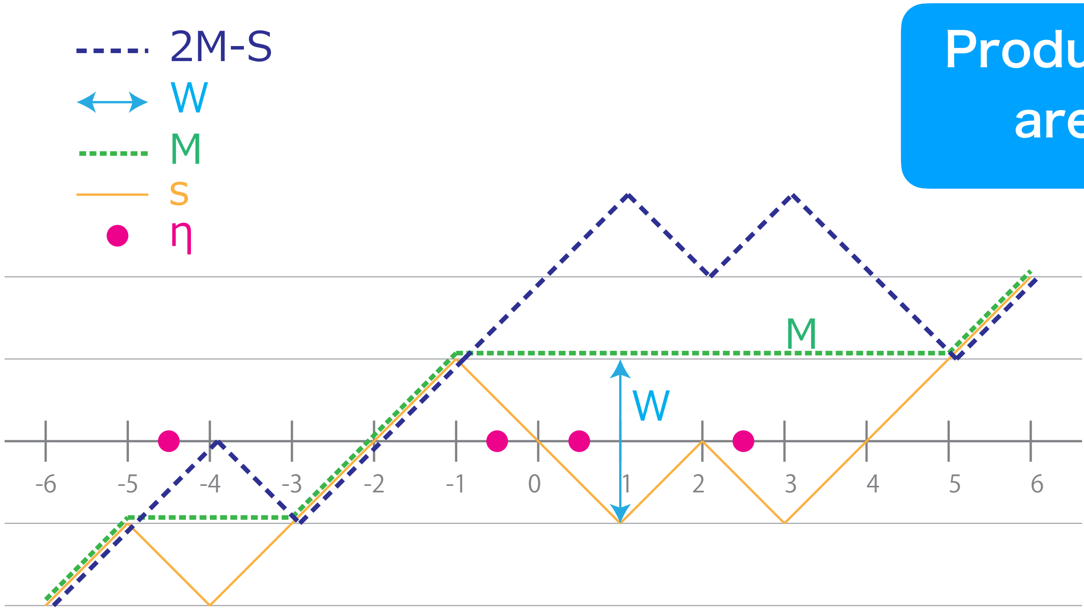


BBS = Pitman's transform

$M_n := \max_{m \le n} S_m$

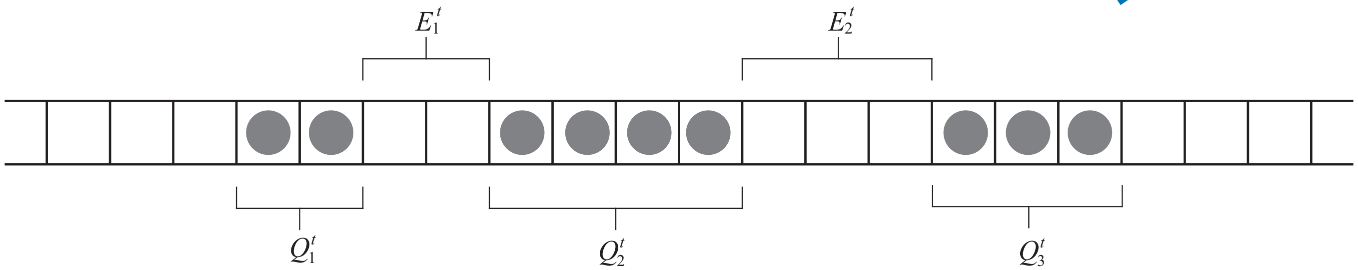
$TS :=$ Path encoding of $T\eta$

Product Bernoulli measures are invariant for BBS !



We have $W_n = M_n - S_n, \quad TS_n = 2M_n - S_n - 2M_0$

Another formulation of BBS (Toda-type)



Dynamics of the BBS :

$$\begin{cases} Q_n^{t+1} = \min\{E_n^t,\ \sum_{j=1}^{n} Q_j^t - \sum_{j=1}^{n-1} Q_j^{t+1}\}, \\ E_n^{t+1} = Q_{n+1}^t + E_n^t - Q_n^{t+1} \end{cases}$$

Let $W_n^t = \sum_{j=1}^{n+1} Q_j^t - \sum_{j=1}^{n} Q_j^{t+1}$ **:**

$$\begin{cases} Q_n^{t+1} = \min\{E_n^t, W_{n-1}^t\}, \\ E_n^{t+1} = Q_{n+1}^t + E_n^t - Q_n^{t+1}, \\ W_n^t = Q_{n+1}^t + W_{n-1}^t - Q_n^{t+1}. \end{cases}$$

$$(Q_n^{t+1}, E_n^{t+1}, W_n^t) = F_{\mathrm{BBS,Toda}}(Q_{n+1}^t, E_n^t, W_{n-1}^t)$$

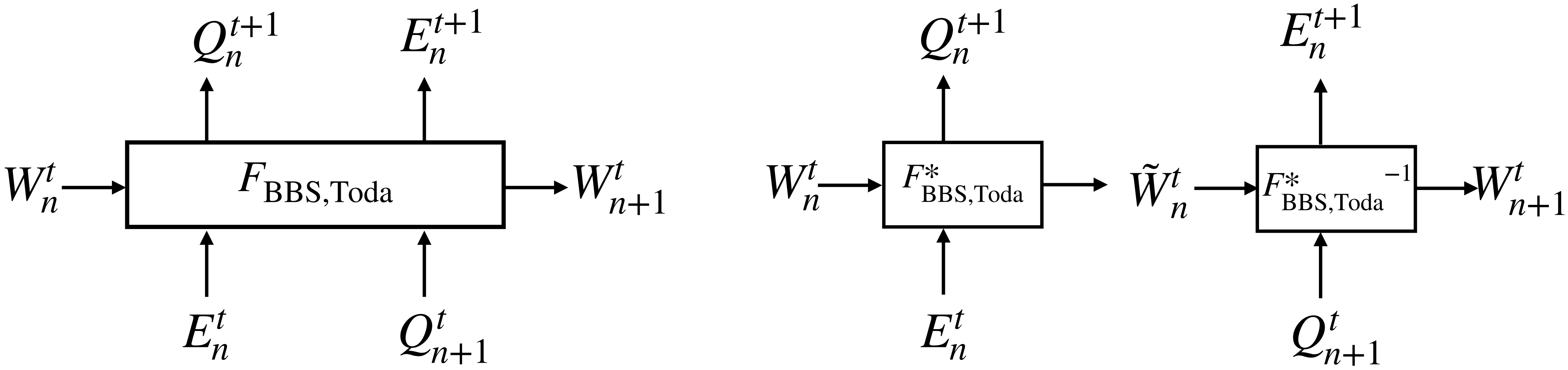
$$F_{\mathrm{BBS,Toda}}(a,b,c) = (\min\{b,c\}, a+b-\min\{b,c\}, a+c-\min\{b,c\})$$

Decomposition of $F_{\text{BBS,Toda}}$ into $F^*_{\text{BBS,Toda}}$

$$F_{\text{BBS,Toda}}(a, b, c) = (\min\{b, c\}, a + b - \min\{b, c\}, a + c - \min\{b, c\})$$

$$F^*_{\text{BBS,Toda}}(a, b) = (\min\{a, b\}, a - b)$$

$$F^{*\ \ -1}_{\text{BBS,Toda}}(a, b) = (a + \max\{0,\ b\}, a + \max\{0, -b\})$$

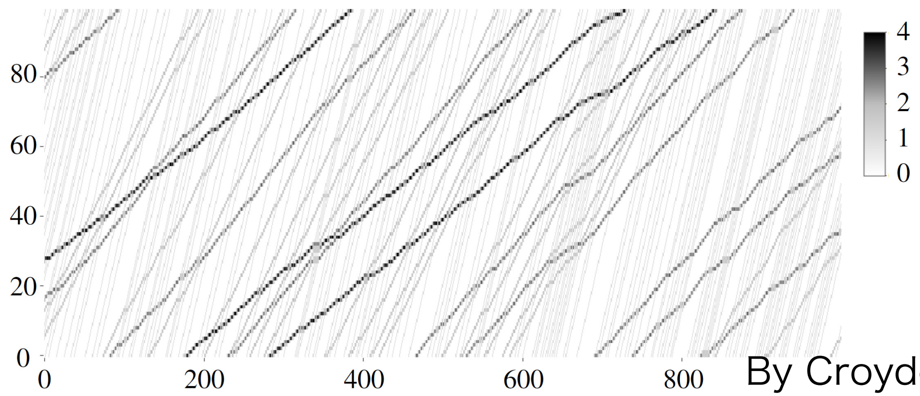


Problems from a probabilistic perspective

- Scaling limits (LLN, fluctuation, LDP) of the soliton distributions for a specific class of random initial measures on $\{0,1\}^N$ as $N \to \infty$. (Levine-Lyu-Pike 2022, Kuniba-Lyu-Okado 2018, Kuniba-Lyu 2019,…)

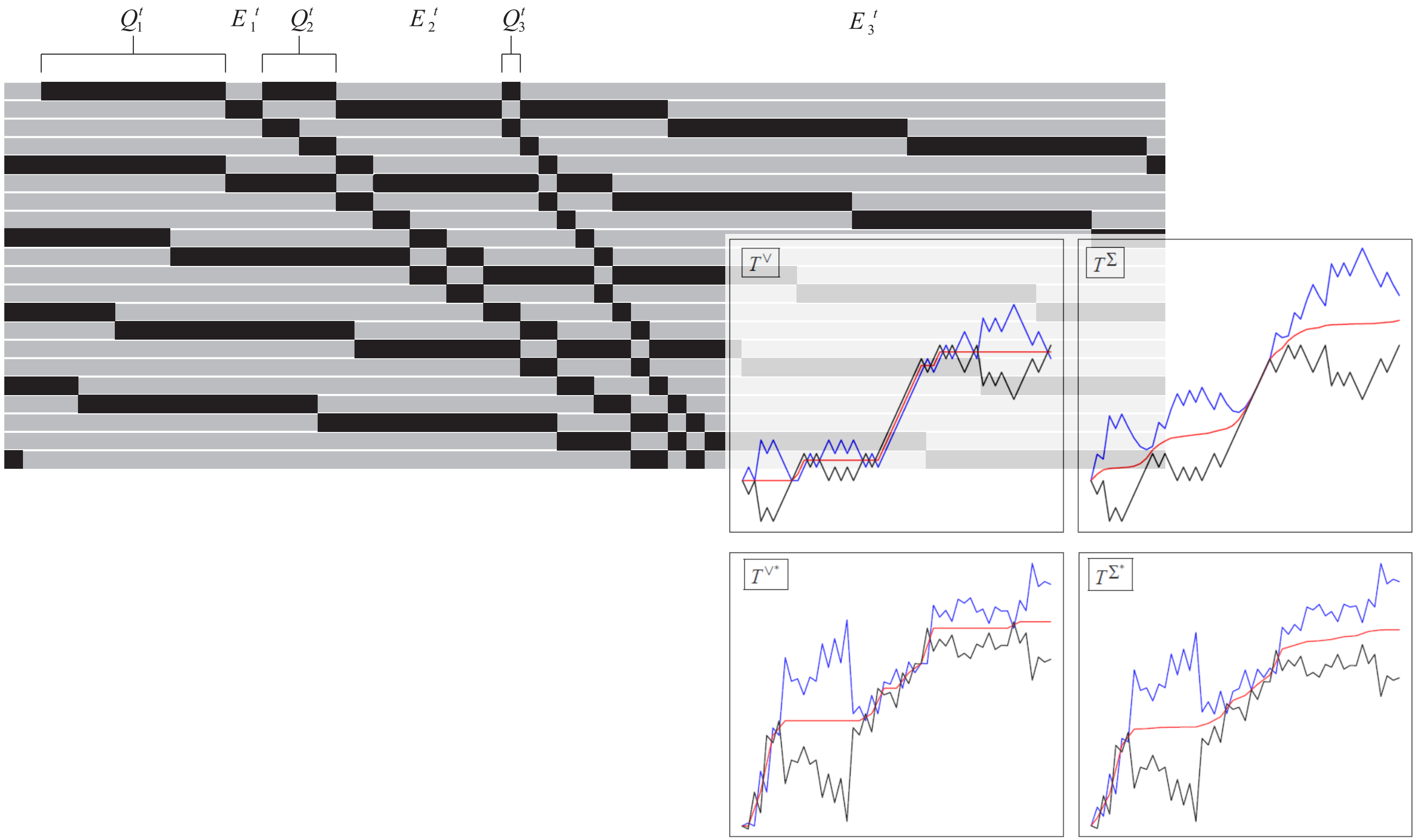
- Construction and characterization of invariant measures. Ergodicity.
  (Ferrari-Nguyen-Rolla-Wang 2021, Croydon-Kato-S-Tsujimoto 2023, Croydon-S 2019, Ferrari-Gabrielli 2020,…)

- Scaling limits of the density of solitons/current/tagged soliton/tagged ball for random initial conditions. (Ferrari-Nguyen-Rolla-Wang 2021, Croydon-Kato-S-Tsujimoto 2023, Croydon-S 2021, Olla-S-Suda 2024+)



By Croydon

2. Other discrete integrable systems



Other discrete integrable models

Korteweg–de Vries (KdV) equation

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0$$

Toda equation

$$\begin{cases} \partial_t I_n = V_n - V_{n-1} \\ \partial_t V_n = V_n(I_n - I_{n+1}) \end{cases}$$

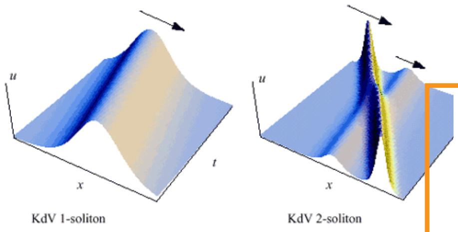


Figure : Brunelli

Discretization

discrete integrable system

Discrete KdV equation

Discrete Toda equation

Ultra-discretization

Ultra-discrete KdV equation

Box-ball system

Ultra-discrete Toda equation

crystallization

Six vertex model (solvable lattice model)

Ultra-discrete KdV (udKdV) equation

$L \in \mathbb{R}$: **Model parameter** , $\eta_n^t \in \mathbb{R}$

$$\eta_n^{t+1} = \min\left\{ L - \eta_n^t \ , \ \sum_{m=-\infty}^{n-1} (\eta_m^t - \eta_m^{t+1}) \right\} \leftrightarrow \ (\eta_n^{t+1}, W_n^t) = F_{\mathrm{udK}}^{(L)}(\eta_n^t, W_{n-1}^t)$$

$$F_{\mathrm{udK}}^{(L)}(a,b) = (\min\{L-a,b\}, a+b-\min\{L-a,b\}), \quad F_{\mathrm{udK}}^{(L)} = {F_{\mathrm{udK}}^{(L)}}^{-1}$$

Path encoding and Pitman's transform for udKdV equation

$$S_n - S_{n-1} := L - 2\eta_n, \quad M_n := \max_{m \le n} \frac{S_m + S_{m-1}}{2}$$

If $L \in \mathbb{N}$ and $\eta_n^t \in \{0,1,2,\ldots,L\}$, then the udKdV equation is the BBS with box capacity L.

Discrete KdV (dKdV) equation

$\delta > 0$: Model parameter , $u_n^t > 0$

$$u_n^{t+1} = \frac{\delta}{u_n^t} + \prod_{m=-\infty}^{n-1} \frac{u_m^t}{u_m^{t+1}} \leftrightarrow (u_n^{t+1}, W_n^t) = F_{\mathrm{dK}}^{(\delta)}(u_n^t, W_{n-1}^t)$$

$$F_{\mathrm{dK}}^{(\delta)}(a, b) = \left(\frac{b}{1 + \delta ab}, a(1 + \delta ab) \right), \quad F_{\mathrm{dK}}^{(\delta)} = F_{\mathrm{dK}}^{(\delta)^{-1}}$$

Path encoding and Pitman's transform for dKdV equation

$$S_n - S_{n-1} := -\log \delta - 2 \log u_n, \quad M_n := \log \left(\sum_{m \le n} \exp \left(\frac{S_n + S_{n-1}}{2} \right) \right)$$

Remarks

- DKdV equation has a close from without W nor the infinite product.

$$u_n^{t+1} = \frac{\delta}{u_n^t} + \prod_{m=-\infty}^{n-1} \frac{u_m^t}{u_m^{t+1}} \leftrightarrow \frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} = \delta(u_n^{t+1} - u_{n+1}^t)$$

- UdKdV equation is the ultra-discretization of dKdV equation.

$$(+,\times) \to (\min,+)$$

$$u_n^{t+1} = \frac{\delta}{u_n^t} + \prod_{m=-\infty}^{n-1} \frac{u_m^t}{u_m^{t+1}} \to \eta_n^{t+1} = \min\{L - \eta_n^t, \sum_{m=-\infty}^{n-1} (\eta_m^t - \eta_m^{t+1})\}$$

- DKdV equation is a discretization of KdV equation. Precisely, KdV equation is a continuous limit of dKdV equation.

Ultra-discrete Toda (udToda) equation

$Q_n^t, E_n^t \in \mathbb{R}$

$$\begin{cases} Q_n^{t+1} & = \min\{E_n^t,\ \sum_{j=-\infty}^{n} Q_j^t - \sum_{j=-\infty}^{n-1} Q_j^{t+1}\}, \\ E_n^{t+1} & = Q_{n+1}^t + E_n^t - Q_n^{t+1} \end{cases} \quad \leftrightarrow$$

$$(Q_n^{t+1}, E_n^{t+1}, W_n^t) = F_{\mathrm{udT}}(Q_{n+1}^t, E_n^t, W_{n-1}^t)$$

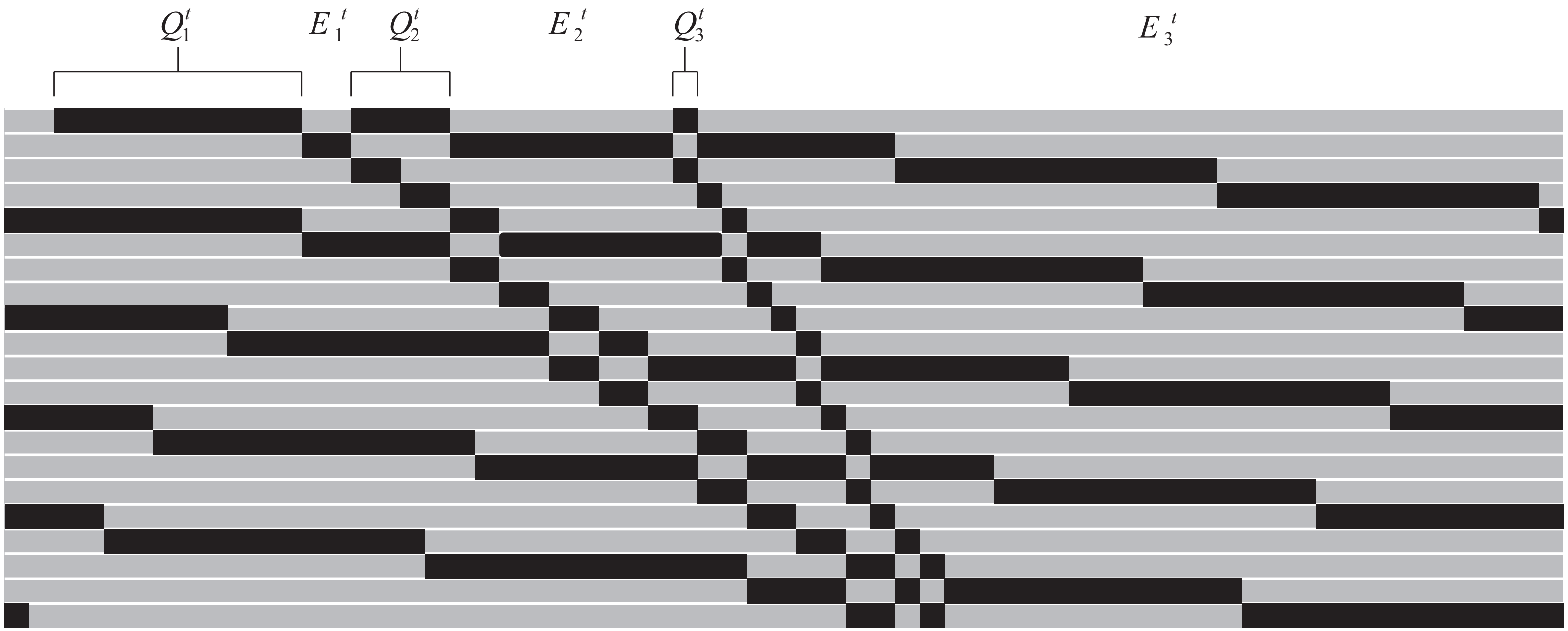
$$F_{\mathrm{udT}}(a,b,c) = (\min\{b,c\}, a+b-\min\{b,c\}, a+c-\min\{b,c\})$$

$$F^*_{\mathrm{udT}}(a,b) = (\min\{a,b\}, a-b)$$

Path encoding and Pitman's transform for udToda equation

$$S_{2n+1} - S_{2n} := -Q_{n+1}, \quad S_{2n} - S_{2n-1} := E_n, \quad M_{2n+1} := \max_{m \le n} S_{2m}, \quad M_{2n} = \frac{M_{2n-1} + M_{2n+1}}{2}$$

Ultra-discrete Toda equation



If $Q_n^t, E_n^t \in \mathbb{N}$, then the udToda equation is the BBS.

Discrete Toda (udToda) equation

$$I_n^t > 0,\ V_n^t > 0$$

$$\begin{cases} I_n^{t+1} = V_n^t + \dfrac{\prod_{j=-\infty}^{n} I_j^t}{\prod_{j=-\infty}^{n-1} I_j^{t+1}} \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases} \leftrightarrow \begin{cases} I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1} \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases} \leftrightarrow (I_n^{t+1}, V_n^{t+1}, W_n^t) = F_{\mathrm{dT}}(I_{n+1}^t, V_n^t, W_{n-1}^t)$$

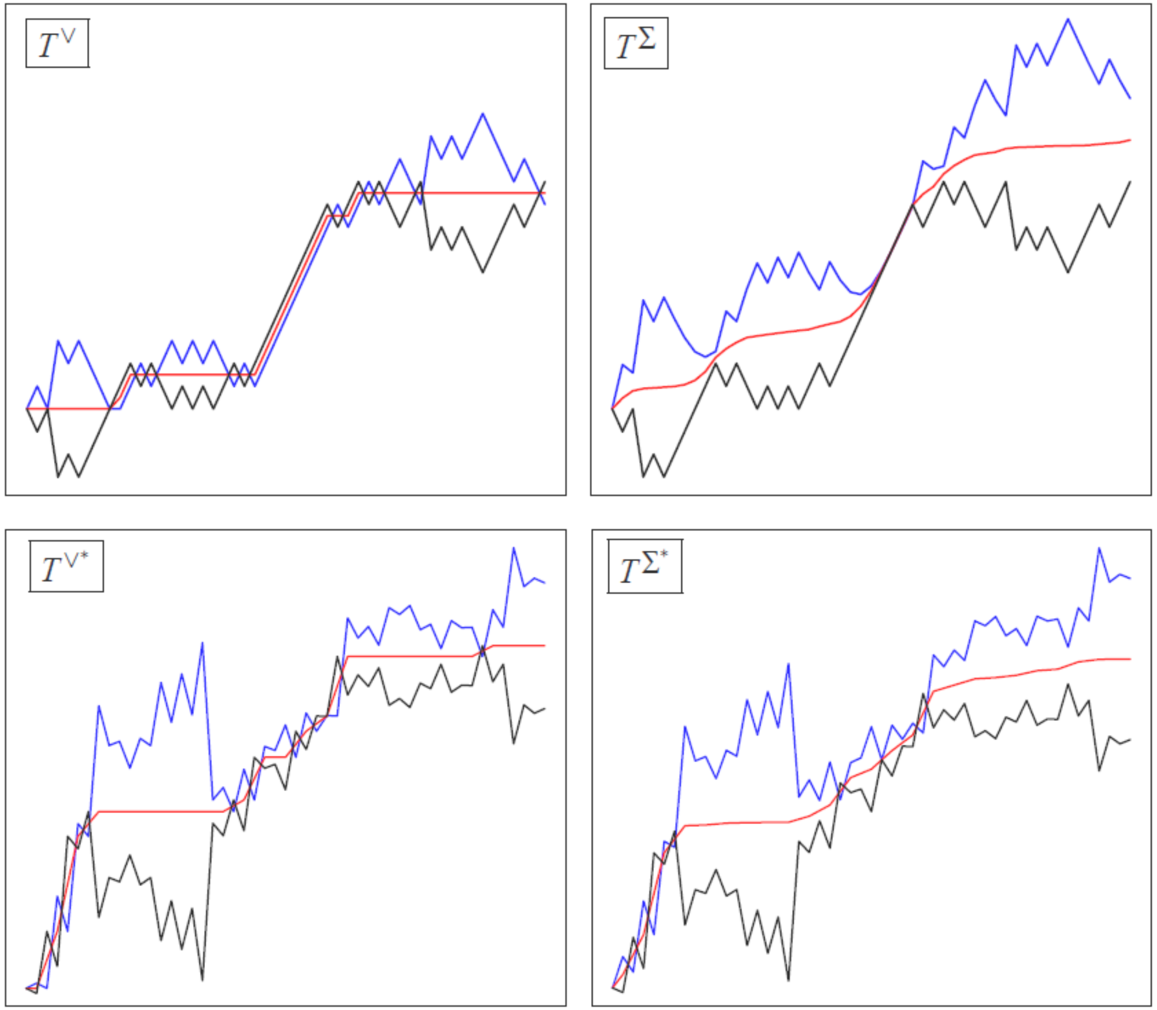
$$F_{\mathrm{dT}}(a,b,c) = \left(b+c, \frac{ab}{b+c}, \frac{ac}{b+c} \right) \qquad F_{\mathrm{dT}}^*(a,b) = \left(a+b, \frac{a}{a+b} \right)$$

Path encoding and Pitman's transform for udToda equation

$$S_{2n+1} - S_{2n} := \log I_{n+1}, \quad S_{2n} - S_{2n-1} := -\log V_n$$

$$M_{2n+1} := \log\Big(\sum_{m \le n} \exp(S_{2m})\Big), \quad M_{2n} = \frac{M_{2n-1} + M_{2n+1}}{2}$$

Pitman's type transforms



By Croydon

Results for d/ud KdV/Toda equations (Croydon-S-Tsujimoto 2022)

- Formalize the bi-infinite dynamics as a solution of initial value problem with the 2-dimensional lattice description

$$\begin{cases} x_n^0 = x_n \quad \forall n \in \mathbb{Z} \\ (x_n^{t+1}, y_n^t) = F(x_n^t, y_{n-1}^t) \quad \forall n, t \in \mathbb{Z} \end{cases}$$

- Introduce a path encoding and derive a Pitman's type transform description $S \to TS = 2M - S - 2M_0$

- Prove that all Pitman's type transforms are well-defined and invariant on asymptotically linear functions with positive drift :

$$\mathcal{S}^{\mathrm{lin}} := \{S : \mathbb{Z} \to \mathbb{R} \mid \exists \lim_{n \to \pm\infty} \frac{S_n}{n} > 0,\ S_0 = 0\}$$

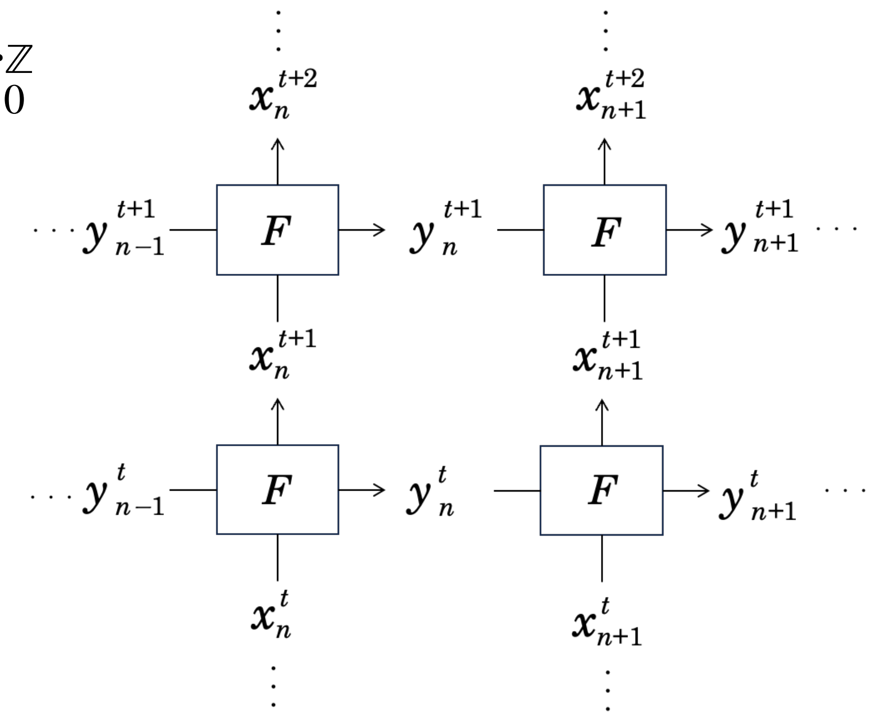
- Moreover, the existence and uniqueness of solution holds on $\mathcal{S}^{\mathrm{lin}}$. The set of configurations whose path-encoding is in $\mathcal{S}^{\mathrm{lin}}$ includes the support of many shift ergodic measures.

3. General frameworks and theorems for invariant measures

KdV-type locally-defined dynamics

$\mathscr{X}_0, \mathscr{Y}_0$: **Polish spaces** $\quad \mathscr{X} := \mathscr{X}_0^{\mathbb{Z}}$

$$F : \mathscr{X}_0 \times \mathscr{Y}_0 \to \mathscr{X}_0 \times \mathscr{Y}_0, \quad F = F^{-1}$$



For a given $x = (x_n) \in \mathscr{X}$, $(x_n^t, y_n^t)_{n,t \in \mathbb{Z}}$ **is**

a solution of the initial value problem

for x **if**

$$\begin{cases} x_n^0 = x_n & \forall n \in \mathbb{Z} \\ (x_n^{t+1}, y_n^t) = F(x_n^t, y_{n-1}^t) & \forall n, t \in \mathbb{Z} \end{cases}$$

$\mathscr{X}^* := \{x \in \mathscr{X}$; $\exists!$ solution of initial value problem for $x\}$

Characterization of i.i.d. type invariant measures

For $x \in \mathcal{X}^*$, $Tx := x^1 = (x_n^1) \in \mathcal{X}$ is well-defined where (x_n^t, y_n^t) is the unique solution of the initial value problem for x.

Theorem (Croydon-S, 2021)

Let μ be a probability measure on $\mathcal{X}_0$ satisfying $\mu^{\mathbb{Z}}(\mathcal{X}^*) = 1$. Then,

$\mu^{\mathbb{Z}} = T\mu^{\mathbb{Z}}$ (i.e. $\mu^{\mathbb{Z}}$ is invariant)

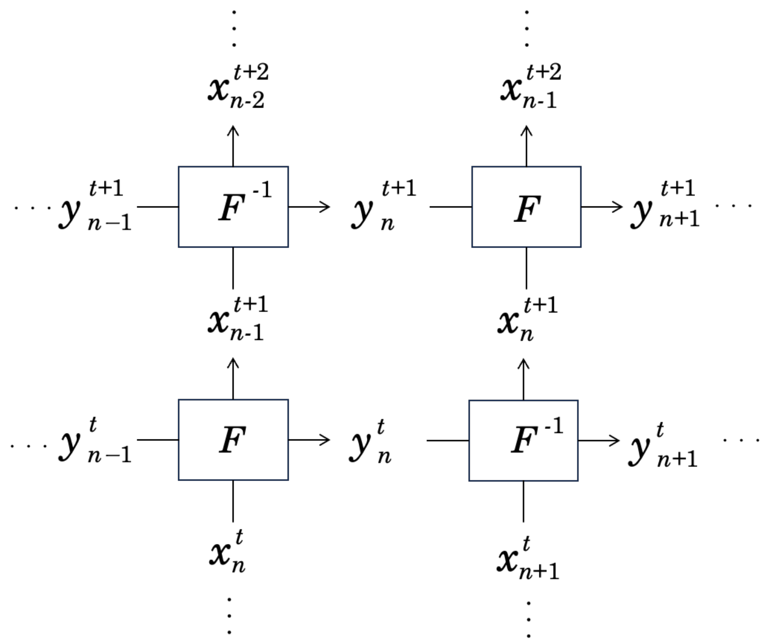
$\Leftrightarrow$ $\exists \nu$: a probability measure on $\mathcal{Y}_0$ such that $F(\mu \times \nu) = \mu \times \nu$

Toda-type locally-defined dynamics

$\mathscr{X}_0, \tilde{\mathscr{X}}_0, \mathscr{Y}_0, \tilde{\mathscr{Y}}_0$: **Polish spaces** $\quad \mathscr{X} := (\mathscr{X}_0 \times \tilde{\mathscr{X}}_0)^{\mathbb{Z}}$

$F : \mathscr{X}_0 \times \mathscr{Y}_0 \to \tilde{\mathscr{X}}_0 \times \tilde{\mathscr{Y}}_0, \quad F$: **bijection**

$$F_{2n} := F, \quad F_{2n+1} := F^{-1}$$



For a given $x = (x_n) \in \mathscr{X}$, $(x_n^t, y_n^t)_{n,t\in\mathbb{Z}}$ **is a solution of the initial value problem for** x **if**

$$\begin{cases} x_n^0 = x_n & \forall n \in \mathbb{Z} \\ (x_{n-1}^{t+1}, y_n^t) = F_n(x_n^t, y_{n-1}^t) & \forall n, t \in \mathbb{Z} \end{cases}$$

$\mathscr{X}^* := \{x \in \mathscr{X} \ ; \ \exists!$ solution of initial value problem for $x\}$

Characterization of i.i.d. type invariant measures

Theorem (Croydon-S, 2021)

Let $\mu, \tilde{\mu}$ be probability measures on $\mathscr{X}_0, \tilde{\mathscr{X}}_0$ satisfying $(\mu \times \tilde{\mu})^{\mathbb{Z}}(\mathscr{X}^*) = 1$.

Then,

$(\mu \times \tilde{\mu})^{\mathbb{Z}} = T(\mu \times \tilde{\mu})^{\mathbb{Z}}$ **(i.e. $(\mu \times \tilde{\mu})^{\mathbb{Z}}$ is invariant)**

$\Leftrightarrow$ $\exists \nu, \tilde{\nu}$: probability measures on $\mathscr{Y}_0, \tilde{\mathscr{Y}}_0$ such that $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$

Independence preserving property

Theorem (Kac 1939, Bernstein 1941)

For $F(x,y)=(x+y,x-y)$. $F(\mu\times\nu)=\tilde{\mu}\times\tilde{\nu} \;\Leftrightarrow\; \mu=\mathrm{N}(a,\sigma),\ \nu=\mathrm{N}(b,\sigma)$

Theorem (Lukacs 1955)

For $F(x,y)=\left(x+y,\dfrac{x}{x+y}\right)$. $F(\mu\times\nu)=\tilde{\mu}\times\tilde{\nu} \;\Leftrightarrow\; \mu=\mathrm{Gam}(a,\sigma),\ \nu=\mathrm{Gam}(b,\sigma)$

Theorem (Ferguson 1964,1965, Crawford 1966)

For $F(x,y)=(\min\{x,y\},x-y)$.

$F(\mu\times\nu)=\tilde{\mu}\times\tilde{\nu} \;\Leftrightarrow\; \mu=\mathrm{sExp}(a,\sigma),\ \nu=\mathrm{sExp}(b,\sigma)$ or

$\mu=\mathrm{ssGeo}(a,m,\sigma),\ \nu=\mathrm{ssGeo}(b,m,\sigma)$

Independence preserving property

Theorem (Matsumoto-Yor, 2000)

For $F(x,y) = \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y}\right)$.

$F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu} \iff \mu = \mathrm{GIG}(a, b, -\sigma),\ \nu = \mathrm{Gam}(b, \sigma)$

Theorem (Croydon-S 2021 (if), Letac-Wesołowski 2022 (only if))

For $F^{(\alpha,\beta)}(x,y) = \left(y\frac{1+\beta xy}{1+\alpha xy}, x\frac{1+\alpha xy}{1+\beta xy}\right)$.

$F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu} \iff \mu = \mathrm{GIG}(a, b\alpha, \sigma),\ \nu = \mathrm{GIG}(a, b\beta, \sigma)$

Application to concrete models

✦ For $p \in \left(0, \frac{1}{2}\right)$, $F_{\mathrm{BBS}}\left(\mathrm{Ber}(p) \times \mathrm{Geo}\left(\frac{1-2p}{1-p}\right)\right) = \mathrm{Ber}(p) \times \mathrm{Geo}\left(\frac{1-2p}{1-p}\right)$. Hence,

$\eta = (\eta_n) : \mathrm{Ber}(p)$ **i.i.d.** is invariant for the BBS.

✦By Crawford's theorem, $(Q_n)_n : \mathrm{Geo}(1 - q_1 q_2)$ **i.i.d.** $(E_n)_n : \mathrm{Geo}(1 - q_1)$ **i.i.d.** is invariant for the BBS.

✦By Crawford's theorem, $(Q_n)_n : \mathrm{Exp}(\lambda_1)$ **i.i.d.** $(E_n)_n : \mathrm{Exp}(\lambda_2)$ **i.i.d.**, $\lambda_1 < \lambda_2$ is invariant for the udToda equation.

✦By Lukacs's theorem $(I_n)_n : \mathrm{Gam}(\lambda_1, \sigma)$ **i.i.d.** $(V_n)_n : \mathrm{Gam}(\lambda_2, \sigma)$ **i.i.d.**, $\lambda_1 > \lambda_2$ is invariant for the dToda equation.

✦By Matsumoto-Yor's theorem, $x = (x_n) : \mathrm{GIG}(c, c\delta, \sigma)$ **i.i.d.** is invariant for the dKdV equation with parameter δ.

Yang-Baxter maps and independence preserving property

Integrability $? \Leftrightarrow ?$ Existence of i.i.d. invariant measures

dmKdV equation : $F^{(\alpha,\beta)}(x,y) = \left(\frac{y(1+\beta xy)}{1+\alpha xy}, \frac{x(1+\alpha xy)}{1+\beta xy} \right), \quad F_{\mathrm{dK}}^{(\delta)} = F^{(\delta,0)}$

Yang-Baxter map : $F_{12}^{(\alpha,\beta)} \circ F_{13}^{(\alpha,\gamma)} \circ F_{23}^{(\beta,\gamma)} = F_{23}^{(\beta,\gamma)} \circ F_{13}^{(\alpha,\gamma)} \circ F_{12}^{(\alpha,\beta)} : \mathbb{R}_+^3 \to \mathbb{R}_+^3$

Independence preserving property :

$$F^{(\alpha,\beta)}(\mathrm{GIG}(\lambda, a\alpha, b) \times \mathrm{GIG}(\lambda, b\beta, a)) = \mathrm{GIG}(\lambda, b\alpha, a) \times \mathrm{GIG}(\lambda, a\beta, b)$$

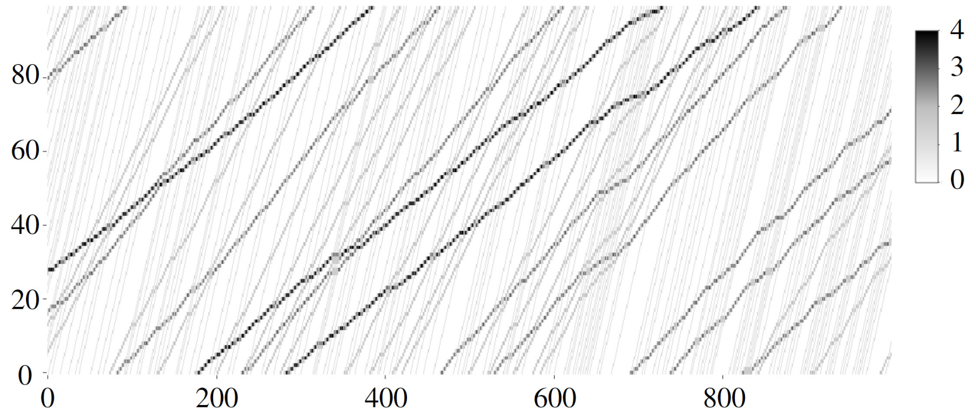
S-Uozumi (2022) : New classes of functions satisfying IP property are found in Yang-Baxter maps. A special class of YB maps leads most of functions having IP property by change of variables/limiting procedure.

Multivariate version? New integrable models?

4. Generalized Hydrodynamics for BBS

Generalized hydrodynamics (GHD) for BBS

- Generalized Gibbs Ensembles (GGE) is characterized by the density of solitons $\rho = (\rho_k)_{k\in\mathbb{N}}$
- Under the GGE with a soliton density $\rho = (\rho_k)_{k\in\mathbb{N}}$, the speed of size k soliton is $v_k^{\mathrm{eff}}(\rho) = v_k - \sum_{m\in\mathbb{N}} \kappa(k,m)\rho_m(v_m^{\mathrm{eff}}(\rho) - v_k^{\mathrm{eff}}(\rho))$ with $v_k = k,\ \kappa(k,m) = 2\min\{k,m\}$.
- In non-equilibrium, the density of solitons $\rho(t) = (\rho(t,k))_{k\in\mathbf{N}}$ evolves according to the GHD equation : $\partial_t\rho_k(t,u) + \partial_u(v_k^{\mathrm{eff}}(\rho(t,u))\rho_k(t,u)) = 0$



Rigorous results for BBS

- There is a "nice class" of invariant measures which are uniquely characterized by the density of solitons $\rho = (\rho_k)_{k \in \mathbb{N}}$ (Ferrari-Gabrielli, 2020)

- For a class of nice invariant measures with the soliton density $\rho = (\rho_k)_{k \in \mathbb{N}}$, the speed of size k soliton satisfies $v_k^{\mathbf{eff}}(\rho) = v_k - \sum_{m \in \mathbb{N}} \kappa(k,m)\rho_m(v_m^{\mathbf{eff}}(\rho) - v_k^{\mathbf{eff}}(\rho))$ (Ferrari-Nguyen-Rolla-Wang 2021)

- In non-equilibrium, the density of solitons $\rho(t) = (\rho(t,k))_{k \in \mathbf{N}}$ evolves according to the GHD equation : $\partial_t \rho_k(t,u) + \partial_u (v_k^{\mathbf{eff}}(\rho(t,u)\rho_k(t,u)) = 0$ (Croydon-S, 2021) (Under several assumptions)

Final comments

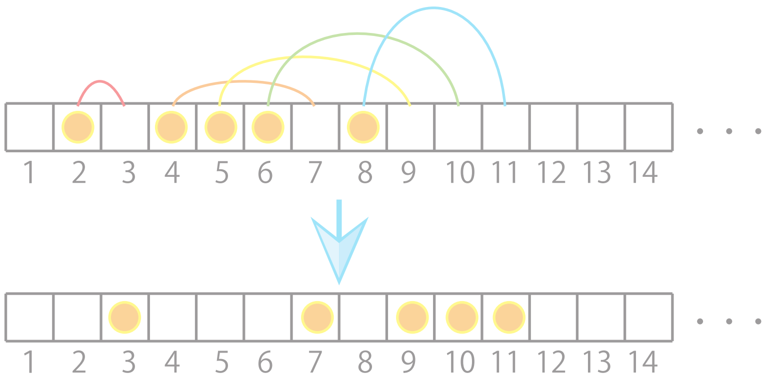
★ Independence preserving property are also essential for some stochastic integrable models.

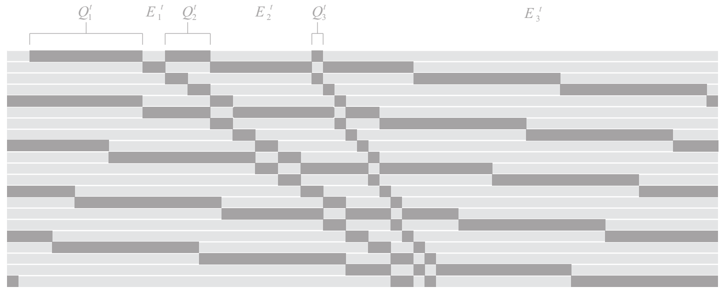
★ Invariant measures (generalized Gibbs measures) for Toda and discrete Toda equations are related to random matrices via Lax matrices.

★There are many open problems :

- For some stochastic integrable models (called integrable Markov processes), it is shown that the transition probability solves an integrable differential equation. How about discrete models?
- For a continuous path S, Pitman's transform determines the continuous BBS. How to characterize solitons/invariant measures?
- Non i.i.d. invariant measures for models other than BBS? GHD for models other than BBS?
- CLT and LDP for the soliton density of BBS?

Discrete integrable systems remain a rich source of hidden mathematical wonders!





Thank you very much for your attention!