The stochastic quantisation of the fractional Φ^4 model in the full subcritical domain

Abstract

I will present a (sketch) of the stochastic quantisation of a family of subcritical (i.e. superrenormalizable) scalar Euclidean QFT via the flow equation method of Duch. Euclidean QFT are measures on distributional fields which should be considered natural generalisation of Markov processes in higher dimension and which play a fundamental role in the rigorous construction of relativistic quantum fields. Stochastic quantisation is a method to realise such measures as pushforward of Gaussian measures via maps obtained by solving PDEs with random sources. In the last 10 years our understanding of the stochastic quantisation method has progressed greatly giving us new methods to attach the problems of EQFTs. The aim of the minicourse is to present, in most of the details, the various aspects of the construction of a particular class of EQFTs showcasing how probabilistic arguments merge with PDE estimates and renormalization group ideas.

`Version 0.1 (16/5/2025). Use these notes with caution, they were not carefully proofread!`

Based on the paper arXiv:2303.18112. Website https://trimester2025.math.gssi.it/gubinelli/

1 Introduction

All along fix $d=3$. Let $\varepsilon>0$, $M=\varepsilon N$ for some $N\in\mathbb{N}$ and $\mathbb{R}_\varepsilon=\varepsilon\mathbb{Z}$. Let $\mathbb{T}^d_{\varepsilon,M}=(\mathbb{R}_\varepsilon/M\mathbb{Z})^d$ the ε-discrete periodic torus of lenght M. Consider the family of measures $\nu_{M,\varepsilon}$ on $\mathbb{R}^{\mathbb{T}^d_{\varepsilon,M}}$ defined by

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi):=\frac{\exp\left(-\int_{\mathbb{T}^d_{\varepsilon,M}}v_\varepsilon(\varphi(x))\mathrm{d}x\right)}{Z_{M,\varepsilon}}\mu_{M,\varepsilon}(\mathrm{d}\varphi) \tag{1}$$

where $\mu_{M,\varepsilon}$ is the M centered Gaussian field with covariance

$$(m^2+(-\Delta_{M,\varepsilon})^s)^{-1}$$

where $(-\Delta_{M,\varepsilon})^s$ is the periodic fractional Laplacian with $s\in(3/4,1)$ on $\mathbb{T}^d_{\varepsilon,M}$ and $v_\varepsilon:\mathbb{R}\to\mathbb{R}$ is the even polynomial

$$v_\varepsilon(\phi)=\lambda\phi^4-c_\varepsilon\phi^2$$

where $\lambda>0$ is a fixed parameter.

We use the notation $\int_{\mathbb{T}^d_{\varepsilon,M}}\mathrm{d}x$ to mean the counting measure $\varepsilon^d\sum_{x\in\mathbb{T}^d_{\varepsilon,M}}$.

The goal of these lectures is to prove the following theorem

Theorem 1. *For any $s\in(3/4,1)$, there is a choice of $(c_\varepsilon)_\varepsilon$ such that the family of measure $\nu_{\varepsilon,M}$ is tight and any accumulation point is a probability measure on $\mathcal{S}'(\mathbb{R}^3)$ enjoying the following properties: translation invariance, reflection positivity and stretched exponential integrability of certain distributional seminorms.*

Translation invariance and reflection positivity will follow for free from their discrete equivalent for the Gaussian measures $\mu_{\varepsilon,M}$ and their local Gibbsian perturbations, like (1).

Our main task is to obtain distributional estimates which are good enought to have tighness of the family of measures $(\nu_{\varepsilon,M})_{\varepsilon,M}$. More precisely each $\nu_{\varepsilon,M}$ describes the law of a random field φ defined on the discrete set $\mathbb{R}^d_\varepsilon$ and then extended on the whole $\mathbb{R}^d$ in some convenient way as a distribution $\mathcal{E}\varphi\in\mathcal{S}'(\mathbb{R}^d)$. For example we could set

$$\mathcal{E}\varphi=\sum_y\varphi(y)\delta_y$$

where δ_y is a Dirac distribution centered in y. This extension is easy but not quite regular enough and we will use another one replacing δ_y by a smoother distribution.

Ignoring for the moment this particular problem, we will focus on getting good estimates of the measures $\nu_{\varepsilon,M}$ and we will do so by studying a particular diffusion $\phi(t)$ which has $\nu_{\varepsilon,M}$ as marginal measure. In particular, $\nu_{\varepsilon,M}$ is the invariant measure of the Langevin SDE

$$\partial_t \phi(t,x) = (m^2 + (-\Delta)^s)\phi(t,x) - v_\varepsilon'(\phi(t,x)) + 2^{1/2}\xi(t,x), \qquad t\in\mathbb{R}, x\in\mathbb{T}^d_{\varepsilon,M}. \tag{2}$$

where ξ is a i.i.d. family, indexed by $x\in\mathbb{T}^d_{\varepsilon,M}$, of white noises in the "time" variable $t\in\mathbb{R}$. We assume that this equation has a stationary solution in t such that, for all $t\in\mathbb{R}$,

$$\mathrm{Law}(\phi(t)) = \nu_{\varepsilon,M}. \tag{3}$$

There are various standard ways to prove this. Let us stress that we do not need uniqueness of the invariant measure, nor ergodicity, in order to run our argument. All that it is needed is the existence of a stationary solution satisfying (3).

It is convenient to see both random fields $\xi(t,x)$ and $\phi(t,x)$ as a M-periodic functions defined on all $\mathbb{R}^d_\varepsilon$.

2 Apriori estimates

In this section we focus on deriving *apriori* estimates for classical solutions φ to the PDE

$$\partial_t\varphi = (-\Delta)^s\varphi - \lambda\varphi^3 + f \tag{4}$$

on $\mathbb{R}\times\mathbb{R}^d_\varepsilon$ with a source term $f\in C(\mathbb{R}\times\mathbb{R}^d_\varepsilon)$.

Before starting, let us give some more details on the fractional Laplacian. For $s\in(0,1)$, the fractional Laplacian $(-\Delta)^s$ acts on (bounded) functions $f\colon\mathbb{R}^d_\varepsilon\to\mathbb{R}$ as

$$(-\Delta)^s f(x) = \int_0^\infty ((1-e^{\Delta\lambda})f)(x)\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}} \tag{5}$$

where $e^{\Delta\lambda}$ is the discrete heat kernel on $\mathbb{R}^d_\varepsilon$. Using Fourier transform we have, for $f,g\in\mathcal{S}(\mathbb{R}^d_\varepsilon)$,

$$\left\langle g, \int_0^\infty ((1-e^{\Delta\lambda})f)\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}}\right\rangle = \int_{(\mathbb{R}^d_\varepsilon)^*}\frac{\mathrm{d}\xi}{(2\pi)^d}\hat{g}(\xi)\int_0^\infty((1-e^{-|\xi|^2\lambda})f)\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}}\hat{f}(\xi)$$

we note then that, by scaling the integration variable λ, we obtain for any $\xi\neq 0$,

$$\int_0^\infty (1-e^{-|\xi|^2\lambda})\frac{\mathrm{d}\lambda}{\lambda^{1+s}} = \int_0^\infty (1-e^{-\lambda})\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}}|\xi|^{2s}$$

so it is enough to take

$$C_s = \int_0^\infty (1-e^{-\lambda})\frac{\mathrm{d}\lambda}{\lambda^{1+s}} < +\infty,$$

to show that

$$\left\langle g, \int_0^\infty ((1-e^{\Delta\lambda})f)\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}}\right\rangle = \langle g, (-\Delta)^s f\rangle.$$

Using the discrete heat kernel $e^{\Delta\lambda}(x)$ one derives an expression for the fractional Laplacian as an integral operator (e.g. on bounded functions)

$$(-\Delta)^s f(x) = \varepsilon^d\sum_y k_s(x-y)[f(x)-f(y)],$$

where the (discrete) integral kernel k_s is positive, given by

$$k_s(y) = \int_0^\infty e^{\Delta\lambda}(y)\frac{\mathrm{d}\lambda}{C_s\lambda^{1+s}}, \qquad y\in\mathbb{R}^d_\varepsilon, |y|\neq 0.$$

Using standard estimates for the discrete heat kernel

$$e^{\Delta\lambda}(x) \lesssim \lambda^{-d/2} e^{-|x|^2/\lambda},$$

one derives the bound

$$|k_s(y)| \lesssim \int_0^\infty e^{-|y|^2/\lambda} \frac{\mathrm{d}\lambda}{\lambda^{1+s+d/2}} \lesssim \int_0^\infty e^{-1/\lambda} \frac{\mathrm{d}\lambda}{\lambda^{1+s+d/2}} \lesssim |y|^{-d-2s}, \qquad y \in \mathbb{R}^d_\varepsilon \backslash \{0\},$$

uniformly in ε.

As we see the fractional Laplacian is extremely non-local, with algebraic decay. This is the cause of some nuisances in our analysis.

The formula (5) together with the fact that the heat kernel $e^{\lambda\Delta}$ is a probability measure and therefore Jensen's inequality holds give, for any convex function $\Phi: \mathbb{R} \to \mathbb{R}$,

$$\Phi(f)(x) - (e^{\Delta\lambda}\Phi(f))(x) \leqslant \Phi(f(x)) - \Phi((e^{\Delta\lambda} f)(x)) \leqslant \Phi'(f(x))(f(x) - (e^{\Delta\lambda} f)(x))$$

where Φ' is a subdifferential for Φ, i.e. it satisfies

$$\Phi(b) - \Phi(a) \geqslant \Phi'(a)(b-a), \qquad a, b \in \mathbb{R}.$$

As a consequence we have a chain inequality for convex function

$$(-\Delta)^s \Phi(f)(x) = \int_0^\infty ((1 - e^{\Delta\lambda})\Phi(f))(x) \frac{\mathrm{d}\lambda}{C_s \lambda^{1+s}} \leqslant \Phi'(f(x))((-\Delta)^s f)(x). \tag{6}$$

Note that if $\Phi' \geqslant 0$ and $v, w \in \mathbb{R}^m$ are two vectors, we have

$$\Phi(|v|^2) - \Phi(|w|^2) \geqslant \Phi'(|w|^2)(|v|^2 - |w|^2) = \Phi'(|w|^2)(2w \cdot (v-w) + |v-w|^2) \geqslant 2\Phi'(|w|^2)[w \cdot (v-w)]$$

so in particular for scalar function f:

$$(-\Delta)^s \Phi(|f|^2) \leqslant 2\Phi'(|f|^2) f((-\Delta)^s f) \tag{7}$$

and a similar inequality holds for vector valued fuctions.

Assume to start that $\|\varphi\|, \|f\| < +\infty$ where $\|\|$ denotes the L^∞ norm over the space-time domain Λ_ε.

In order to estimate Eq. (4) we use a space-time weight

$$q(t,x) = (1 + \ell|x|^2 + \ell|t|^2)^{-K}$$

for some large K and consider the convex function $\Phi(x) = (x - L^2)_+$ with $\Phi'(x) = \mathbb{I}_{x > L^2}$ and test the equation against $q\Phi'(\varphi)\varphi$ to get

$$\int q\, \Phi'(\varphi^2)\varphi(\partial_t \varphi + (-\Delta)^s \varphi + \lambda\varphi^3) = \int q\, \Phi'(\varphi)\varphi f$$

where the integral is over the space-time $\Lambda_\varepsilon = \mathbb{R} \times \mathbb{R}^d$. The contribution of the time derivative is integrated by parts (eventually approximating Φ with a smooth function) to give

$$\int q\, \Phi'(\varphi^2)\varphi \partial_t \varphi = \frac{1}{2} \int q\, \partial_t \Phi(\varphi^2) = -\frac{1}{2} \int \partial_t q\, \Phi(\varphi^2) \geqslant -\frac{1}{2} \left\| \frac{\partial_t q}{q} \right\| \int q \Phi(\varphi^2)$$

while for that of the fractional Laplacian we use (7) to have

$$\int q\, \Phi'(\varphi^2)\varphi((-\Delta)^s \varphi) \geqslant \frac{1}{2} \int q\, (-\Delta)^s \Phi(\varphi^2) = \frac{1}{2} \int ((-\Delta)^s q) \Phi(\varphi^2) \geqslant -\frac{1}{2} \left\| \frac{(-\Delta)^s q}{q} \right\| \int q \Phi(\varphi^2).$$

Finally, the cubic nonlinearity gives

$$\int q\, \Phi'(\varphi^2)\varphi^4 = \int q\, \mathbb{I}_{\varphi^2 > L^2} \varphi^4 \geqslant L^2 \int q\, \mathbb{I}_{\varphi^2 > L^2} [(\varphi^2 - L^2) + L^2] \geqslant L^2 \int q\, (\varphi^2 - L^2)_+ + L^4 \int q\, \mathbb{I}_{\varphi^2 > L^2}$$

so overall we have

$$\lambda L^2 \int q \Phi(\varphi^2) + \lambda L^4 \int q \Phi'(\varphi^2) \leqslant \underbrace{\frac{1}{2} \left(\left\| \frac{\partial_t q}{q} \right\| + \left\| \frac{(-\Delta)^s q}{q} \right\| \right)}_{=:Q} \int q \Phi(\varphi^2) + \|f\| \, \|\varphi\| \int q \Phi'(\varphi^2).$$

Due to the choice of the weight q we have

$$Q := \frac{1}{2}\left(\left\|\frac{\partial_t q}{q}\right\| + \left\|\frac{(-\Delta)^s q}{q}\right\|\right) < +\infty$$

this is easy to see for the time derivative and requires some more estimate for the fractional Laplacian, e.g. using the integral representation.

Choosing L large enough (ignoring the dependence on $\lambda > 0$), i.e.

$$L \gtrsim \max\left(Q^{1/2}, (\|f\|\|\varphi\|)^{1/4}\right)$$

we conclude that

$$\int q\Phi'(\varphi) = \int q\Phi(\varphi) = 0,$$

that is $\varphi^2 \leqslant L^2$ since $q > 0$ everywhere. Therefore

$$\|\varphi\| \lesssim \max\left[Q^{1/2}, \|f\|^{1/4}\|\varphi\|^{1/4}\right].$$

Via Young's inequality we can bound $\|f\|^{1/4}\|\varphi\|^{1/4} \lesssim \varepsilon^{-1}\|f\|^{1/3} + \varepsilon\|\varphi\|^{1/4}$ for some small ε while by a suitable choice of ℓ, Q can be made arbitrarily small. We conclude that

$$\|\varphi\| \lesssim \|f\|^{1/3}.$$

This argument allows a variation were we control a weighted norm of φ and not only $\|\varphi\|$. For this we consider again a weight ρ and let $v := \rho\varphi$. We assume now that

$$\|\rho\varphi\|, \|\rho^3 f\| < +\infty$$

and try to repeat the above argument by testing the equation of φ with $q\Phi'(v^2)\rho^3 v$ and proving the inequalities

$$\int q\,\Phi'(v^2)\,\rho^3\,v\,\partial_t\varphi \geqslant -\int \partial_t(q\rho^2)\Phi(v^2) - \frac{1}{2}\int q\,\Phi'(v^2)\,(\rho\partial_t\rho)\,v^2$$

$$\geqslant -(\|\rho(\partial_t\rho)\| + Q)\int q\,\Phi(v^2) - \underbrace{\frac{1}{2}\|v\|^2\,\|\rho(\partial_t\rho)\|}_{=:B_1}\int q\,\Phi'(v^2).$$

and (after some computations...)

$$\int q\Phi'(v^2)\,v\rho^3\,(-\Delta)^s\varphi^a \geqslant -\left(\frac{1}{2}\|(-\Delta)^s\rho^2\| + Q\right)\int q\Phi(v^2) - B_2\int q\Phi'(v^2)$$

for a constant

$$B = B_1 + B_2 := \frac{1}{2}\|\rho\varphi\|\left(\|\rho(\partial_t\rho)\| + \|\rho(-\Delta)^s\rho\| + \|\rho^2\mathfrak{D}_s(\rho^{-1})\mathfrak{D}_s(\rho)\|\right) + \|\rho\mathfrak{D}_s(\rho)\mathfrak{D}_s(\rho\varphi)\|$$

where we introduced the (non-linear) operator $\mathfrak{D}_s(f)$ defined by

$$\mathfrak{D}_s^2(f)(t,x) := \int_{\mathbb{R}_\varepsilon^d}[f(t,y) - f(t,x)]^2 k_s(x-y) \tag{8}$$

which behaves like a fractional derivative of order s.

The cube satisfies a similar inequality as above,

$$\int q\,\Phi'(v^2)\rho^3\,v\varphi^3 = \int q\,\mathbb{I}_{v^2>L^2}\,v^4 \geqslant L^2\int q\,\mathbb{I}_{v^2>L^2}[(v^2 - L^2) + L^2] \geqslant L^2\int q\,\Phi(v^2) + L^4\int q\,\Phi'(v^2)$$

and for the force we have

$$\int q\,\Phi'(v^2)\rho^3\,v\,f \leqslant \|\rho^3 f\|\,\|v\|\int q\,\Phi'(v^2)$$

Letting

$$A := \frac{1}{2}\|(-\Delta)^s\rho^2\| + \frac{1}{2}\|\rho(\partial_t\rho)\|$$

we conclude that (ignoring Q which can be made small and avoid tracing λ)

$$\|v\| \lesssim \max(A^{1/2}, B^{1/4}, \|\rho^3 f\|^{1/3}).$$

This concludes the apriori estimates.

3 The scale decomposition

The apriori estimates cannot be applied directly to (2) since the equation is too singular: the force is a distribution.

To make our life easier we think to φ, ξ as space M-periodic functions on the space–time domain $\Lambda_\varepsilon = \mathbb{R} \times \mathbb{R}_\varepsilon^d$ and to $(-\Delta)^s$ as the fractional heat operator on the same domain.

We do not actually expect solution to be bounded in weighted L^∞ as $\varepsilon \to 0$ and we need to smooth out the solution in order to use the above apriori estimates.

We introduce a family of smoothing operators $(J_\sigma)_{\sigma \in (0,1]}$ with $J_1 = \mathrm{Id}$ such that they will filter out space-time scales $\lesssim [\![\sigma]\!]$ with $[\![\sigma]\!] = 1 - \sigma$. A natural way to implement them is via Fourier multipliers:

$$J_\sigma f(t,x) = \int_{\Lambda_\varepsilon^*} j_\sigma(|\omega|^{1/2s}) j_\sigma(q_\varepsilon(k)) \hat{f}(\omega,k) e^{i(\omega t + k\cdot x)} \frac{\mathrm{d}\omega \mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x) \in \Lambda_\varepsilon.$$

where $q_\varepsilon(k) \approx |k|$ is the symbol of the square root of the discrete Laplacian:

$$q_\varepsilon(\xi) := \left[\sum_{i=1}^d \left(\frac{1}{\varepsilon}\sin(\varepsilon\,\xi_i)\right)^2\right]^{1/2}, \tag{9}$$

and $j_\sigma(\eta) = j([\![\sigma]\!]\eta/\sigma)$ with j a smooth and compactly supported function with $j(\eta) = 0$ if $|\eta| \geqslant 2$ and $j(\eta) = 1$ if $|\eta| \leqslant 1$.

Intuitively

$$J_\sigma \approx j_\sigma(|\mathcal{L}|^{1/2})$$

where $\mathcal{L} = \partial_t + (-\Delta)^s$.

Define $\phi_\sigma := J_\sigma \varphi$ and note the this is a smooth function, solving the equation (cfr. (2))

$$\mathcal{L}\phi_\sigma = J_\sigma F(\phi), \qquad t \in \mathbb{R}, x \in \mathbb{T}_{\varepsilon,M}^d. \tag{10}$$

where

$$F(\phi) = -\lambda\phi^3 - r_\varepsilon \phi + \xi.$$

Note that ϕ_σ can be represented via the integral formula

$$\phi_\sigma(t) = G_\sigma F(\phi)$$

where $G_\sigma := G J_\sigma$ with G the solution operator for the linear part of the equation (2):

$$(Gf)(t) := \int_{-\infty}^t e^{-(1+(-\Delta)^s)(t-s)} f(s) \mathrm{d}s.$$

Estimate for this equation show that, for $s \in (0,1)$,

$$|G(t,x)| \lesssim \frac{\mathbb{I}_{t \geqslant 0} t e^{-ct}}{(|(t,x)|)^{d+2s}}, \qquad (t,x) \in \Lambda_\varepsilon$$

uniformly in $\varepsilon > 0$ where

$$|z| = |x| + |t|^{1/2s}, \qquad z = (t,x) \in \Lambda_\varepsilon$$

is the distance function which scales correctly with the homogeneity of the fractional heat operator $\partial_t + (-\Delta)^s$.

Note that if $\lambda = 0$ then this is a linear equation

$$\mathcal{L}\varphi_\sigma = J_\sigma \xi, \qquad \varphi_\sigma = G_\sigma \xi$$

and one can show that (for some suitable algebraic weight ρ)

$$\|\rho J_\sigma \xi\|_{L^\infty} \lesssim [\![\sigma]\!]^{-(d+2s)/2}$$

and moreover that

$$\|\rho\varphi_\sigma\| = \|\rho G_\sigma \xi\| \lesssim [\![\sigma]\!]^{2s-(d+2s)/2} \approx [\![\sigma]\!]^{-(d-2s)_+/2}$$

which is the expected regularity of the field φ_σ also in the non-linear setting.

We introduce now a family of *functionals* $(F_\sigma)_\sigma$ such that $F_1 = F$ and use the fundamental theorem of calculus to write

$$F(\phi) = F_\sigma(\phi_\sigma) + \int_\sigma^1 \partial_\eta [F_\eta(\phi_\eta)] \mathrm{d}\eta = F_\sigma(\phi_\sigma) + R_\sigma$$

with

$$R_\sigma = \int_\sigma^1 [(\partial_\eta F_\eta)(\phi_\eta) + \mathrm{D}F_\eta(\phi_\eta)(\partial_\eta \phi_\eta)] \mathrm{d}\eta = \int_\sigma^1 [(\partial_\eta F_\eta)(\phi_\eta) + \mathrm{D}F_\eta(\phi_\eta)\dot{G}_\eta F_\eta(\phi_\eta)] \mathrm{d}\eta + \int_\sigma^1 \mathrm{D}F_\eta(\phi_\eta)\dot{G}_\eta R_\eta \mathrm{d}\eta$$

where $\mathrm{D}F_\eta(\varphi) \cdot \psi$ denotes the Frèchet derivative of F_η in the direction of ψ at the point φ.

Assume now we are able to find solutions to the *flow equation*

$$\partial_\eta F_\eta(\varphi) + \mathrm{D}F_\eta(\varphi)\dot{G}_\eta F_\eta(\varphi) = 0 \tag{11}$$

for every value of φ and $\eta \in (0,1)$ with boundary condition $F_1 = F$, then $R_\sigma = 0$ and ϕ_σ satisfies the PDE:

$$\mathcal{L}\phi_\sigma = F_\sigma(\phi_\sigma).$$

The form of the r.h.s. of this equation depends on the solution F_σ of the flow equation, which we call *effective force* since it describes the source term appearing in the evolution of the effective description ϕ_σ of ϕ at (space–time) scales $\gtrsim [\![\sigma]\!]$.

To undestand what is the form of F_σ, observe that we have

$$F_\sigma(\varphi) = F(\varphi) + \int_\sigma^1 \mathrm{D}F_\eta(\varphi)\dot{G}_\eta F_\eta(\varphi) \mathrm{d}\eta,$$

and since $F(\varphi)$ is polynomial, the solution can be approximated via the Picard iterations of this integral equaiton. The first gives

$$-\lambda\varphi^3 + r_\varepsilon \varphi + \xi + \int_\sigma^1 (-\lambda\varphi^2 + r_\varepsilon)\dot{G}_\eta(-\lambda\varphi^3 + r_\varepsilon\varphi + \xi) \mathrm{d}\eta$$

and more generally each iteration gives rise to a sum of monomials in φ and ξ which are non-local (due to the non-locality of $\dot{G}_\eta$).

Note that $\dot{G}_\sigma$ can be estimated as follows:

$$|\partial^A \dot{G}_\sigma(z)| \lesssim [\![\sigma]\!]^{-d-1-|A|} (1 + (\varepsilon \vee |z|)/[\![\sigma]\!])^{-d} (1 + |z|/[\![\sigma]\!])^{-2s+\varepsilon}, \tag{12}$$

where ∂^A symbolize a multiple space-time derivation and $|A|$ its space-time homogeneity, i.e. the number of space derivatives and $2s$ times the number of time derivatives. Recall that $|z|$ is the fractional parabolic distance $|(t,x)| = |t|^{1/2s} + |x|$.

This means that it possesses some space-time locality at scale $[\![\sigma]\!]$. In the case of the Laplacian the situation is much better as the decay can be made (with suitable choice of j) stretched-exponential in the distance.

We are not going to attempt to solve exactly the flow equation (11). We instead try to find an appropriate approximation by introducting the sequence of functionals

$$F_\sigma^{[0]} = F_1, \qquad F_\sigma^{[\ell+1]} = -\sum_{\ell_1+\ell_2=\ell} \int_\sigma^1 \mathrm{D}F_\eta^{[\ell_1]} \dot{G}_\eta F_\eta^{[\ell_2]} \mathrm{d}\eta.$$

By induction, all have the following general form:

$$F_\sigma^{[\ell]}(\psi)(z) = \sum_{k \geqslant 0} \int_{\Lambda^k} F_\sigma^{[\ell](k)}(z, z_1, \ldots, z_k) \psi(z_1) \cdots \psi(z_k) \mathrm{d}z_1 \cdots \mathrm{d}z_k$$

where the $F_\sigma^{[\ell](k)}(z, z_1, \ldots, z_k)$ are coefficient which are random distributions in all their variables. For example we have

$$F_\sigma^{[0](3)}(z, z_1, \ldots, z_3) = -\lambda \delta(z - z_1) \delta(z - z_2) \delta(z - z_3), \qquad F_\sigma^{[0](1)}(z, z_1) = r_\varepsilon \delta(z - z_1),$$

$$F_\sigma^{[0](0)}(z) = 2^{1/2} \xi(z).$$

Intuitively, the functional $F_\sigma^{[\leqslant \ell]} = F_\sigma^{[0]} + \cdots + F_\sigma^{[\ell]}$ is the ℓ-th Picard iterate for the flow equation (11).

Using this approximation at level ℓ_*, i.e. setting $F_\sigma = F_\sigma^{[\leqslant \ell]}$ we have that R_σ satisfies

$$R_\sigma = \int_\sigma^1 \left[\sum_{\substack{\ell_1+\ell_2 > \ell_* \\ \ell_1, \ell_2 \leqslant \ell_*}} \mathrm{D}F_\eta^{[\ell_1]}(\phi_\eta) \dot{G}_\eta F_\eta^{[\ell_2]}(\phi_\eta) \right] \mathrm{d}\eta + \int_\sigma^1 \mathrm{D}F_\eta(\phi_\eta) \dot{G}_\eta R_\eta \mathrm{d}\eta$$

and

$$\mathcal{L}\phi_\sigma = J_\sigma [F_\sigma(\phi_\sigma) + R_\sigma].$$

4 Sketch of the PDE strategy

For simplicity lets us ignore for the moment the remainder R_σ, and roughly model the original equation with

$$\mathcal{L}\phi_\sigma - \lambda \phi_\sigma^3 \approx F_\sigma(\phi_\sigma) - \lambda \phi_\sigma^3 \tag{13}$$

where $F_\sigma(\phi_\sigma)$ is a polynomial in the field ϕ_σ.

To measure the space-time grown of the fields we use a weight in the form

$$\zeta_\sigma(z) := (1 + [\![\sigma]\!]^{2a} |z|_s^2)^{-1/2}, \qquad z \in \Lambda,$$

where $a > 1$ is an exponent introduced to match the scale behaviour with the large-distance behaviour and whose appropriate choice will be crucial to close our estimates.

Pathwise bounds on the random effective force $(F_\sigma)_\sigma$ are derived via the analysis of its probabilistic cumulants via a flow equation. The result of this analysis, **which we ignore here,** is that the effective force $F_\sigma(\phi_\sigma)$ is a random non-local polynomial of the field ϕ_σ with coefficients which are localized in regions of size $\approx [\![\sigma]\!]$ and which are roughly of size

$$F_\sigma^{[\ell](k)} \approx [\![\sigma]\!]^{(k-3)\beta + \delta\ell}$$

where k is the order of the monomial. Here $\beta > \gamma$ represent the size of the fields in the flow equation analysis, $\delta > 0$ is a measure of the distance to criticality and ℓ the perturbative order. Due to a Kolmogorov-type argument needed to extract the almost sure behaviour of the force F_σ from its moments, we loose also a bit in the space-time growth, which will be modelled by a weight $\zeta_\sigma^{-\kappa_{\mathfrak{o}}(\ell+1)}$ where $\kappa_{\mathfrak{o}} > 0$ is an arbitrarily small exponent. Overall we have, schematically,

$$F_\sigma(\phi_\sigma) \approx \sum_{k, \ell} \zeta_\sigma^{-\kappa_{\mathfrak{o}}(\ell+1)} [\![\sigma]\!]^{(k-3)\beta + \delta\ell} \phi_\sigma^k \tag{14}$$

and in this sum $\lambda\phi_\sigma^3$ is the term with higher degree with coefficients which are not going to zero as $[\![\sigma]\!]\searrow 0$. The sums over k,ℓ are finite and are determined by conditions allowing to solve the remainder equation for R_σ, which for the moment we will ignore.

To estimate the size of the solution to the PDE (13), we introduce a constant $|||\phi|||$ such that

$$|\phi_\sigma(z)| = |(J_\sigma\phi)(z)| \leqslant \zeta_\sigma^{-1}(z)[\![\sigma]\!]^{-\gamma}|||\phi|||, \qquad z\in\Lambda, \tag{15}$$

valid for all $\sigma\geqslant\bar{\mu}$ where $\bar{\mu}$ is a random scale which will be chosen so that $[\![\bar{\mu}]\!]\ll 1$.

When $|z|\approx[\![\sigma]\!]^{-a}$ then the spatial weight $\zeta_\sigma^{-1}(z)$ is of order one and we are describing the distributional nature of the solution, growing like $[\![\sigma]\!]^{-\gamma}$ for some $\gamma>0$ as $[\![\sigma]\!]\searrow 0$. The spatial growth is arbitrary.

When $|z|\gg[\![\sigma]\!]^{-a}$ the spatial growth can be improved as follows. Let $\hat{\mu}\ll\sigma$ such that $|z|\approx[\![\hat{\mu}]\!]^{-a}$ and observe that

$$\phi_\sigma = J_\sigma\phi_{\hat{\mu}}$$

due to the properties of the smoothing operators, therefore now using this "martingale" property we have another estimate

$$|\phi_\sigma(z)|\approx|(J_\sigma\phi_{\hat{\mu}})(z)|\approx|\phi_{\hat{\mu}}(z)|\lesssim(1+[\![\hat{\mu}]\!]^a|z|)[\![\hat{\mu}]\!]^{-\gamma}|||\phi|||\approx|z|^{\gamma/a}|||\phi|||\approx(1+[\![\sigma]\!]^a|z|)^{\gamma/a}[\![\sigma]\!]^{-\gamma}|||\phi|||. \tag{16}$$

As we see we gained a better spatial weight $\zeta_\sigma^{-\gamma/a}(z)$ instead of $\zeta_\sigma^{-1}(z)$, and boosted our initial "local" estimate (15). We see that by choosing a large enough we can modulate the growth to be as small as we like. As we anticipated, this is crucial in closing the non-linear estimates for the PDE.

The behaviour of the kernel sizes in (14) helps us to handle monomials of large order since for $k>3$ we have $[\![\sigma]\!]^{(3-k)\beta+\delta\ell}\ll 1$. This is an effect of subcriticality. However the spatial growth of the monomials ϕ_σ^k is a big issue. Essentially coercive estimates for (13) give

$$|\phi_\sigma|^3 \approx |F_\sigma(\phi_\sigma)-\lambda\phi_\sigma^3| \approx \zeta_\sigma^{-(L+1)\kappa_\mathfrak{o}}\left[[\![\sigma]\!]^{-3\beta+\delta}+[\![\sigma]\!]^{-2\beta+\delta}|\phi_\sigma|+[\![\sigma]\!]^{-\beta+\delta}|\phi_\sigma|^2+\sum_{k\geqslant 3,\ell}[\![\sigma]\!]^{(k-3)\beta+\delta\ell}|\phi_\sigma|^k\right]. \tag{17}$$

as all the lower order diverging contributions are at least in first-order of perturbation theory and where we took the worse spatial growth given by $\zeta_\sigma^{-(L+1)\kappa_\mathfrak{o}}$ with L the largest value of ℓ which we need to consider. Replacing (16) in (17), ignoring the mild-nonlocality of the effective force, we end up with

$$|\phi_\sigma|^3 \approx \zeta_\sigma^{-(L+1)\kappa_\mathfrak{o}}\left[[\![\sigma]\!]^{-3\beta+\delta}+[\![\sigma]\!]^{-2\beta-\gamma+\delta}\zeta_\sigma^{-\gamma/a}|||\phi|||+[\![\sigma]\!]^{-\beta-2\gamma+\delta}\zeta_\sigma^{-2\gamma/a}|||\phi|||^2+\sum_{k\geqslant 3,\ell}[\![\sigma]\!]^{(k-3)\beta-3\gamma+\delta\ell}\zeta_\sigma^{-k\gamma/a}|||\phi|||^k\right],$$

which can be summarized with the rough estimate

$$|\phi_\sigma|^3\approx[\![\sigma]\!]^{-3\beta+\delta}\zeta_\sigma^{-(L+1)\kappa_\mathfrak{o}-K\gamma/a}(1+|||\phi|||)^K, \tag{18}$$

where K is the maximal degree of the monomials and where we used that $\beta>\gamma$ to compensate for the size of the fields with the size of the kernels. Now the constant $|||\phi|||$ can be estimated by

$$|||\phi|||\approx\sup_{\sigma\geqslant\bar{\mu}}\zeta_\sigma^{1/3}[\![\sigma]\!]^\gamma|\phi_\sigma|\approx[\![\sigma]\!]^{\gamma-\beta+\delta/3}\zeta_\sigma^{(1-(L+1)\kappa_\mathfrak{o}-K\gamma/a)/3}(1+|||\phi|||)^{K/3}$$

and choosing $\gamma\leqslant\beta$ such that $\gamma-\beta+\delta/3\geqslant\epsilon>0$ and then a large enough such that $1-(L+1)\kappa_\mathfrak{o}-K\gamma/a\leqslant 0$ we end up with the estimation

$$|||\phi|||\approx[\![\bar{\mu}]\!]^\epsilon(1+|||\phi|||)^{K/3}.$$

It is clear now that by choosing $[\![\bar{\mu}]\!]\ll 1$ we can close this non-linear estimate and obtain $|||\phi|||\approx 1$. Our problem has been transformed in a small-data quasi-linear problem for which uniform estimates are obtained via a continuity argument.

5 Some more details on the weights

[Add more technical points, e.g. limited decay of $\dot{G}_\sigma$???]

Let's discuss the handling of the non-locality of the effective force kernels. The major technical nuisances in this paper are due to the limited decay of the *slice propagator* $\dot{G}_\sigma$ for the fractional parabolic operator $\mathcal{L}$. Roughly speaking we have only an algebraic behaviour of the type (see Lemma ?):

$$|\dot{G}_\sigma(z)| \lesssim [\![\sigma]\!]^{-d-1} \, (1 + |z|_s / [\![\sigma]\!])^{-d-2s+\varepsilon}, \qquad z \in \Lambda, \tag{19}$$

where $\varepsilon > 0$ is a small loss in decay and $|z|_s$ is the fractional parabolic distance on Λ. This is very different from the behaviour of the same operator in the case of the Laplacian where the decay is stretched exponential (cfr. [2, 3]). This is also different from the case of the fractional Laplacian in the usual ("static") probabilistic renormalization group approach to the fractional Φ^4 model [1]. This is due to the limited smoothness of the symbol for the fractional heat operator.

[It would be interesting to devise an alternative strategy to bypass this problem with some other scale decomposition (or an additional localization procedure).]

The major consequence of (19) is a similar algebraic decay for the kernels of the effective force, which are obtained from $\dot{G_\sigma}$ solving a flow equation. The monomials which appear in $F_\sigma(\phi_\sigma)$ have indeed the form

$$F_\sigma(\phi_\sigma)(z) \approx \sum_{k,\ell} \int F_\sigma^{[\ell],(k)}(z, y_1, \ldots, y_k) \prod_{j=1}^{k} \phi_\sigma(y_j) \mathrm{d}y_j$$

where $F_\sigma^{[\ell],(k)}$ are random distributional kernels. Ignoring their distributional nature and thinking of them as bona-fide functions, their spatial non-locality and spatial growth is modelled as (cfr. Definition ?)

$$F_\sigma^{[\ell],(k)}(z, y_1, \ldots, y_k) \approx \zeta_\sigma^{-(\ell+1)\kappa_\mathfrak{o}}(z)(1 + [\![\sigma]\!]^{-1}\mathrm{St}(z, y_1, \ldots, y_k))^{-(\flat - \ell\kappa_\mathfrak{o})}$$

where $\mathrm{St}(z, y_1, \ldots, y_k)$ is a measure of the diameter of the set $\{z, y_1, \ldots, y_k\}$, the initial decay exponent $\flat \approx 2s$ is due to (19) and where the additional loss $\ell\kappa_\mathfrak{o}$ in decay is needed to account for some other losses which intervenese in the solution theory for the effective force F.

It becomes clear that in order to reason as in (18) we need to be able to absorb the spatial growth of the fields ϕ_σ using the limited decay of the kernels. For this reason it is crucial to be able to use the norm $|\!|\!|\phi|\!|\!|$ for which we can "modulate" the spatial decay as in (16):

$$\left| \zeta_\sigma(z) \int F_\sigma^{[\ell],(k)}(z, y_1, \ldots, y_k) \prod_{j=1}^{k} \phi_\sigma(y_j) \mathrm{d}y_j \right| \lesssim |\!|\!|\phi|\!|\!|^k \times$$

$$\times \sup_{(y_i)_i} \zeta_\sigma^{-(\ell+1)\kappa_\mathfrak{o}}(z)(1 + [\![\sigma]\!]^{-1}\mathrm{St}(z, y_1, \ldots, y_k))^{-(\flat - \ell\kappa_\mathfrak{o})} \zeta_\sigma^{-\gamma/a}(y_1) \cdots \zeta_\sigma^{-\gamma/a}(y_k)$$

$$\lesssim |\!|\!|\phi|\!|\!|^k \zeta_\sigma^{-(\ell+1)\kappa_\mathfrak{o} - k\gamma/a}(z) \lesssim |\!|\!|\phi|\!|\!|^k \zeta_\sigma^{-1}(z)$$

where we used the fact that we can choose a large and $\kappa_\mathfrak{o}$ small to have $(\ell+1)\kappa_\mathfrak{o} + k\gamma/a \leqslant 1$ and also guarantee that we can use the weight $(1 + [\![\sigma]\!]^{-1}\mathrm{St}(z, y_1, \ldots, y_k))^{-(\flat - \ell\kappa_\mathfrak{o})}$ to move the weights on the leafs $(y_i)_i$ to the root z.

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