Fast relaxation of a viscous vortex in an external flow

Based on joint work with Martin Donati.

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I Introduction: The incompressible Navier-Stokes eq. in $\mathbb{R}^2$

$u(x,t) = (u_1(x,t), u_2(x,t))$: velocity of the fluid at point $x \in \mathbb{R}^2$ and time $t \geq 0$

$p(x,t)$ = pressure / density

$\nu > 0$ = kinematic viscosity = viscosity/density [$10^{-6}\, m^2/s$ for water]

$$\begin{cases} \partial_t u(x,t) + (u(x,t)\cdot\nabla)u(x,t) = \nu\Delta u(x,t) - \nabla p(x,t) \\ \mathrm{div}\, u(x,t) = 0. \end{cases} \qquad (NS)$$

1^{st} eq: evolution of the momentum $\left(\frac{1}{m}\sum F = a\right)$

2^{nd} eq: incompressibility condition

Trajectories of the fluid particles:

$$\begin{cases} X'(t) = u(X(t),t) \\ X(t_0) = X_0 \in \mathbb{R}^2 \end{cases} \implies X(t) = \Phi_{t,t_0}(X_0).$$

The flow map Φ_{t,t_0} is a volume preserving diffeomorphism of $\mathbb{R}^2$:

$$\det(D\Phi_{t,t_0})(X_0) = \exp\left(\int_{t_0}^{t} \mathrm{div}(u(X(t')),t')\,dt'\right) = 1.$$

- Vorticity formulation $\omega := \partial_1 u_2 - \partial_2 u_1 = \operatorname{curl} u$.

Observe that $(u\cdot\nabla)u = \frac{1}{2}\nabla|u|^2 + u^\perp\omega$, $u^\perp = \begin{pmatrix}-u_2\\ u_1\end{pmatrix}$

$\Rightarrow \operatorname{curl}(u\cdot\nabla)u = \partial_1(u_1\omega) + \partial_2(u_2\omega) = u\cdot\nabla\omega$ since $\operatorname{div} u = 0$.

So taking the curl of (NS) we get the vorticity equation:

$$\partial_t\omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) = \nu\Delta\omega(x,t). \qquad \text{(VE)}$$

Not a closed equation for ω, because (VE) involves the velocity field u.

- Biot-Savart formula To express u in terms of ω, we have to solve:

$$\begin{cases}\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 = 0\\ \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1 = \omega\end{cases}$$

Stream function: $u = \nabla^\perp\psi = \begin{pmatrix}-\partial_2\psi\\ \partial_1\psi\end{pmatrix} \Rightarrow \begin{cases}\operatorname{div} u = 0\\ \Delta\psi = \omega\end{cases}$

Poisson formula:

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,\omega(y)\,dy$$

Taking the curl we thus obtain the Biot-Savart formula: $u = BS[\omega]$

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,dy. \qquad \text{(BS)}$$

(VE) + (BS) is a closed evolution system for the vorticity ω.

⚠ u is a linear but nonlocal function of ω! Incompressible fluid mechanics is always nonlocal.

The calculations so far are formal, see below for sufficient conditions on ω so that (VE), (BS) make sense.

Radially symmetric vortices

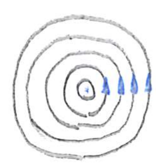
We use polar coordinates : $x = (r\cos\vartheta, r\sin\vartheta)$, and we denote

$$e_r = \frac{x}{|x|}, \quad e_\vartheta = \frac{x^\perp}{|x|}.$$

If $\omega = \omega(|x|)$ is radially symmetric, so is $\psi = \psi(|x|)$, hence

$$u = \nabla^\perp \psi(|x|) = \psi'(|x|)\frac{x^\perp}{|x|} \implies \underline{u = V(|x|)\,e_\vartheta} \qquad (V = \psi')$$

$$\operatorname{div} u = \frac{1}{r}\partial_r(r u_r) + \frac{1}{r}\partial_\vartheta u_\vartheta = 0$$



$$\operatorname{curl} u = \frac{1}{r}\partial_r(r u_\vartheta) - \frac{1}{r}\partial_\vartheta u_r = \frac{1}{r}(rV)'.$$

$$\implies V(r) = \frac{1}{r}\int_0^r s\,\omega(s)\,ds. \qquad \Big\| \qquad p'(r) = \frac{1}{r}V(r)^2 \quad \text{(pressure)}$$

In particular : $u\cdot\nabla\omega = \frac{1}{r}V\,\partial_\vartheta\omega = 0$.

It follows that ω is a radially symmetric solution of the heat equation:

$$\Big[\ \partial_t\omega(x,t) = \nu\Delta\omega(x,t).$$

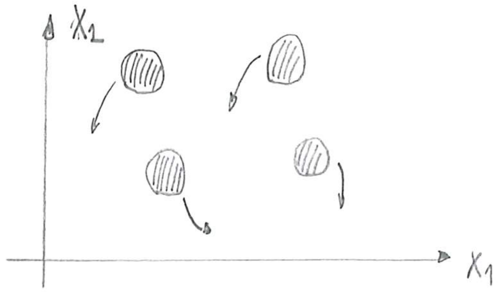
Example 1 : Rankine's vortex

$$\begin{cases} \omega(r) = \dfrac{\Gamma}{\pi R^2}\mathbb{1}_{[0,R]} \implies \Gamma = \displaystyle\int_{\mathbb{R}^2}\omega(|x|)\,dx \\[2ex] V(r) = \dfrac{\Gamma}{2\pi R^2}\begin{cases} r & \text{if } r \le R \quad : \text{rigid rotation} \\ \dfrac{R^2}{r} & \text{if } r \ge R \quad : \text{irrotational flow} \end{cases}, \quad u = \dfrac{\Gamma x^\perp}{2\pi}\begin{cases} \dfrac{1}{R^2} & |x| \le R \\ \dfrac{1}{|x|^2} & |x| \ge R \end{cases} \end{cases}$$

Example 2 : Lamb-Oseen vortex

$$\begin{cases} \omega = \dfrac{\Gamma}{4\pi R^2}e^{-|x|^2/4R^2} \implies \Gamma = \displaystyle\int_{\mathbb{R}^2}\omega(|x|)\,dx \\[2ex] u = \dfrac{\Gamma}{2\pi}\dfrac{x^\perp}{|x|^2}\left(1 - e^{-|x|^2/4R^2}\right), \quad V = \dfrac{\Gamma}{2\pi r}\left(1 - e^{-r^2/4R^2}\right). \end{cases}$$

Interaction of vortices



Consider the interesting situation where the vorticity is a superposition of a finite number of well-separated vortices

$$\omega(x,t) = \sum_{i=1}^{N} \omega_i(x,t) \implies u(x,t) = \sum_{i=1}^{N} u_i(x,t)$$

where $u_i = BS[\omega_i]$ $\forall i \in \{1, \dots, N\}$. The decomposition of ω is not unique, but we postulate that the ω_i satisfy:

$$\left[\partial_t \omega_i + \underset{\substack{\uparrow \\ \text{the full velocity field}}}{u} \cdot \nabla \omega_i = \nu \Delta \omega_i \qquad \forall i \in \{1, \dots, N\} \right. \qquad (*)$$

In particular, given a decomposition of ω at initial time $t = 0$, eq. $(*)$ univoquely defines ω_i at later times. We can rewrite $(*)$ in the equivalent form:

$$\partial_t \omega_i + (u_i + f_i) \cdot \nabla \omega_i = \nu \Delta \omega_i \qquad \forall i \in \{1, \dots, N\} \qquad (**)$$

where $u_i = BS[\omega_i]$ and $f_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} BS[\omega_j]$.

In $(**)$, the term $u_i \cdot \nabla \omega_i$ describes the self-interaction of the i^{th} vortex, and the term $f_i \cdot \nabla \omega_i$ the interaction with the other vortices.

In what follows, to study the interactions of vortices, we consider eq. $(**)$ for a given velocity field f_i satisfying appropriate estimates, and we study the behavior of a single vortex ω_i. This simplified model keeps all important features of the general case, and is somewhat easier to study.

Phenomenology of vortex interactions

If the vortices $\omega_i(x,t)$ have a definite sign, we can consider the center of vorticity $x_i(t)$ given by:

$$x_i(t) = \frac{1}{\Gamma_i} \int_{\mathbb{R}^2} x\, \omega_i(x,t)\, dx, \quad \text{where } \Gamma_i = \int_{\mathbb{R}^2} \omega_i(x,t)\, dx.$$

Using (**) a direct calculation shows that Γ_i is a conserved quantity, whereas

$$x_i'(t) = \frac{1}{\Gamma_i} \int_{\mathbb{R}^2} f_i(x,t)\, \omega_i(x,t)\, dx. \qquad \left\{ \begin{array}{l} \text{Vortices move due to mutual} \\ \text{interactions, not self-interaction!} \end{array} \right.$$

In the (formal) limit where $\omega_i(x,t) \longrightarrow \Gamma_i \delta(x - x_i(t))$, we find:

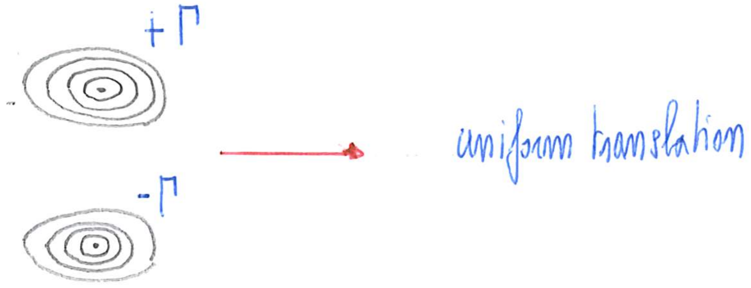
$$x_i'(t) = f_i(x_i(t), t) = \sum_{j \neq i} BS[\omega_j](x_i(t), t)$$

$$= \sum_{j \neq i} \frac{\Gamma_j}{2\pi} \frac{(x_i(t) - x_j(t))^\perp}{|x_i(t) - x_j(t)|^2} \qquad \text{Point vortex system} \quad (PV)$$

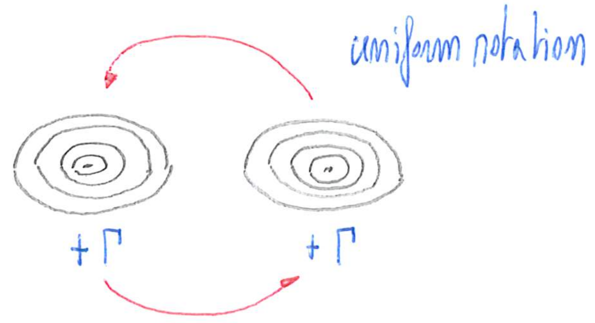
N.B. The argument above, based on the center of vorticity, is the only reasonable one that explains why we should disregard the self-interaction of vortices when deriving (PV).

Beyond the point vortex approximation, one may want to compute the deformation of vortices under the shear stress created by the other vortices:

- Vortex dipole



- Corotating vortex pair



II A viscous vortex in an external flow

From now on we consider the following vorticity equation:

$$\partial_t \omega(x,t) + \big(u(x,t) + f(x,t)\big)\cdot\nabla\omega(x,t) = \nu\Delta\omega(x,t) \qquad \begin{matrix} x\in\mathbb{R}^2 \\ t\geq 0\end{matrix} \qquad (Vf)$$

where: • $u = BS[\omega]$ is the velocity field associated with ω.

• f is a given external field.

We assume that:

• $\operatorname{div} f = \partial_1 f_1 + \partial_2 f_2 = 0$. (essential!)

• f is smooth: $f \in C_b^\infty(\mathbb{R}^2\times[0,T], \mathbb{R}^2)$ (can be relaxed)

In the particular case $f=0$ we recover the usual vorticity eq. (VE). The following results are obtained by adapting the theory for NS to the modified eq. (Vf).

Proposition 1: The Cauchy problem for (Vf) is globally well posed in $L^1(\mathbb{R}^2)$.

For any $\omega_0 \in L^1(\mathbb{R}^2)$, there exists a unique mild solution

$$\omega \in C^0([0,T], L^1(\mathbb{R}^2)) \cap C^0((0,T], L^\infty(\mathbb{R}^2))$$

with $\omega(0) = \omega_0$. This solution is a locally Lipschitz function of $\omega_0 \in L^1(\mathbb{R}^2)$.

The integral eq. associated with (Vf) is:

$$\begin{cases} \omega(t) = S(\nu t)\omega_0 - \displaystyle\int_0^t S(\nu(t-s))\operatorname{div}\big((u(s)+f(s))\omega(s)\big)\,ds, \\ S(t) = e^{t\Delta} : \text{ heat kernel in } \mathbb{R}^2 \end{cases}$$

Prop. 1 can be proved following the same lines as in Ben-Artzi 1993, where the case $f\equiv 0$ is considered.

Conserved quantity: $\Gamma := \int_{\mathbb{R}^2} \omega(x,t)\,dx$ (total circulation)

We are interested in ω describing a vortex, so we assume henceforth that $\Gamma \neq 0$, for instance $\Gamma > 0$.

⚠ This implies that $u \notin L^2(\mathbb{R}^2)$, because of slow decay at infinity!

⇒ Exercise: If $u \in L^2(\mathbb{R}^2)^2$ and $\omega := \partial_1 u_2 - \partial_2 u_1 \in L^1(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2} \omega\,dx = 0$.

If we assume further that $(1+|x|)\omega_0 \in L^1(\mathbb{R}^2)$, the solution of (V_f) satisfies $(1+|x|)\omega(\cdot,t) \in L^1$ $\forall t \geq 0$ and one can define the center of vorticity:

$$\bar{x}(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\,\omega(x,t)\,dx\,, \qquad t \in [0,T].$$

Lemma: $\bar{x}'(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} f(x,t)\,\omega(x,t)\,dx, \quad t \in [0,T].$

Proof: For $t>0$ the solution is smooth and we can compute

$$\bar{x}'(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\,\partial_t \omega(x,t)\,dx$$

$$= \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\Big(\nu\Delta\omega(x,t) - \big(u(x,t)+f(x,t)\big)\cdot\nabla\omega(x,t)\Big)\,dx\,.$$

But:

- $\int x\,\Delta\omega\,dx = 0$ (int. by parts)
- $-\int x\,(u+f)\cdot\nabla\omega\,dx = -\int x\,\mathrm{div}\big((u+f)\omega\big)\,dx = \int (u+f)\,\omega\,dx$
- $\int u(x)\,\omega(x)\,dx = \frac{1}{2\pi}\iint \underbrace{\frac{(x-y)^\perp}{|x-y|^2}}_{\text{odd}}\ \underbrace{\omega(y)\,\omega(x)}_{\text{even}}\,dx\,dy = 0\,.$

So $\bar{x}'(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} f(x,t)\,\omega(x,t)\,dx$ for $t>0$, and the result holds for $t=0$ too by continuity. □

Finite measures as initial data

Let $\mathcal{M}(\mathbb{R}^2)$ be the space of all finite, signed Radon measures on $\mathbb{R}^2$.
If $\mu \in \mathcal{M}(\mathbb{R}^2)$, we have the Jordan decomposition

$$\mu = \mu_+ - \mu_- \qquad (\mu_\pm \text{ are "minimal" positive measures})$$

and the total variation measure $|\mu| = \mu_+ + \mu_-$. We denote

$$\|\mu\|_{TV} = |\mu|(\mathbb{R}^2) \qquad (\text{total variation norm}).$$

Equipped with $\|\cdot\|_{TV}$, $\mathcal{M}(\mathbb{R}^2)$ is a Banach space which contains $L^1(\mathbb{R}^2)$ as a closed subspace.

Relevant examples for fluid mechanics:

i) Absolutely continuous measures: $\mu = \omega\, dx$, $\omega \in L^1(\mathbb{R}^2)$
Ex: vortex patch $\quad \omega = \mathbb{1}_\Omega$, $\Omega \subset \mathbb{R}^2$ smooth and bounded

ii) Singularly continuous measures: Ex: vortex sheet

$$\langle \mu, \varphi \rangle = \int_C \varphi \, d\ell, \qquad C \subset \mathbb{R}^2 \text{ smooth curve}$$

iii) Point vortices: $\quad \mu = \sum_{i=1}^{N} \Gamma_i \, \delta_{x_i}$

It is known that the Cauchy problem for (VE) with $f \equiv 0$ is globally well-posed in the space $\mathcal{M}(\mathbb{R}^2)$. If $\|\mu\|_{TV} \le C_* \nu$: fixed point argument!

- existence can be proved by regularizing the initial data
  - Cottet 1986
  - Giga-Miyakawa-Osada 1988
  - Kato 1994
- uniqueness requires a special argument to deal with large dirac masses:
  - Gallay-Wayne 2005
  - Gallagher-Gallay 2005
  - Bedrossian-Masmoudi 2014

Example: (still with $f \equiv 0$) Lamb-Oseen vortex

If $\omega_0 = \Gamma \delta_0$ the unique solution of (VE) is

$$\omega(x,t) = \frac{\Gamma}{\nu t}\, \Omega_0\left(\frac{x}{\sqrt{\nu t}}\right), \qquad u(x,t) = \frac{\Gamma}{\sqrt{\nu t}}\, U_0\left(\frac{x}{\sqrt{\nu t}}\right) \tag{LO}$$

where

$$\Omega_0(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \qquad U_0(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2}\left(1 - e^{-|\xi|^2/4}\right).$$

We now return to the equation (Vf) with $f \neq 0$. The lower order term $f \cdot \nabla \omega$ in (Vf) does not change the nature of the equation, and we expect that:

Proposition 2: The Cauchy problem for (Vf) is globally well-posed in $\mathcal{M}(\mathbb{R}^2)$.

Exercise (for a good master student or a young PhD student): prove Proposition 2!

In what follows we assume that (Vf) has a unique global solution if

$$\omega_0 = \Gamma \delta_{z_0}, \quad \text{for some } \Gamma \in \mathbb{R} \text{ and } z_0 \in \mathbb{R}^2$$

Why this choice? The external field creates a shear stress that deforms the vortex $\omega(x,t)$ solution of (Vf). In particular, radially symmetric vortices such as (LO) are not possible. Starting with a point vortex allows us to see the deformations occur gradually, since the initial vortex has zero extension. In that sense the initial data above are well-prepared. In contrast, starting with a radially symmetric vortex of nonzero extension gives rise to a time-layer near $t = 0$ where the vortex undergoes damped oscillations to adapt its shape to the external strain; we speak of ill-prepared initial data in such a case. See the last chapter IV for a discussion of this situation.

Relevant physical parameters (assume $\Gamma > 0$)

- Circulation Reynolds number : $Re = \dfrac{\Gamma}{\nu}$.

  We assume henceforth that $Re \gg 1$ and we set $\delta = \dfrac{1}{Re} = \dfrac{\nu}{\Gamma}$.

  The assumption $\delta \ll 1$ ensures that the vortex moves over a long distance (if $f \neq 0$) before spreading due to viscosity.

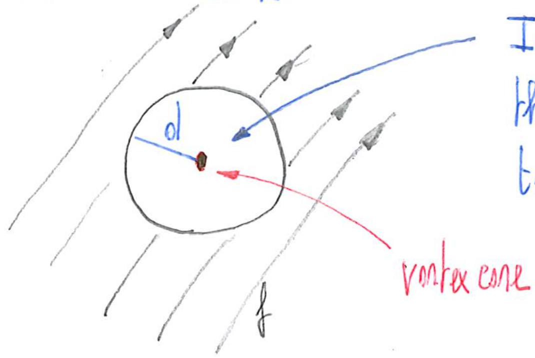
- Characteristic time : $\dfrac{1}{T_0} = \sup_{0 \le t \le T} \sup_{x \in \mathbb{R}^2} |Df(x,t)|$ — Jacobian matrix

  Nearby trajectories of the ODE $z' = f(z,t)$ separate at rate $O(e^{t/T_0})$.

- Effective size of the vortex: $d = \sqrt{\Gamma T_0}$

  The velocity field of a vortex of circulation $\Gamma > 0$ located at the origin behaves like $u = \dfrac{\Gamma}{2\pi} \dfrac{x^\perp}{|x|^2}$ as $|x| \to \infty$. The corresponding strain satisfies:

  $|Du(x)| \approx \dfrac{\Gamma}{2\pi |x|^2} = \dfrac{1}{2\pi T_0}$ when $|x| = d$.



Inside the disk the strain of the velocity field is essentially due to the vortex.

- Aspect ratio : $\varepsilon(t) = \dfrac{\sqrt{\nu t}}{d} = \dfrac{\text{size of the vortex core}}{\text{effective size of the vortex}}$

Important relation:

$$\varepsilon(t)^2 = \frac{\nu t}{d^2} = \frac{\nu t}{\Gamma T_0} = \delta \frac{t}{T_0}.$$

$\Longrightarrow$ As long as t is comparable to T_0 we have $\varepsilon(t)^2 \approx \delta \ll 1$.

Theorem 1: Fix $\Gamma > 0$ and $z_0 \in \mathbb{R}^2$. There exist $K_0 > 0$ and $\delta_0 > 0$ such that, if $0 < \nu < \Gamma\delta_0$, the unique solution of (Vf) with initial data $\omega_0 = \Gamma\delta_{z_0}$ satisfies:

$$\frac{1}{\Gamma}\int_{\mathbb{R}^2}\left|\omega(x,t) - \frac{\Gamma}{\nu t}\Omega_0\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right| dx \leq K_0\,\frac{\sqrt{\nu t}}{d}\,, \quad 0 < t \leq T$$

where $z(t)$ is the solution of the ODE:

$$z'(t) = f(z(t),t)\,, \quad z(0) = z_0\,.$$

For a <u>fixed $\Gamma > 0$</u>, the large constant K_0 and the small constant $\delta_0 > 0$ only depend on the ratio T/T_0 and on the bounds of f and its derivatives.

In particular, the result holds uniformly in ν provided $0 < \delta < \delta_0$.

⚠ The formulation above is not appropriate to study the limit $\Gamma \to 0$ or $\Gamma \to +\infty$!

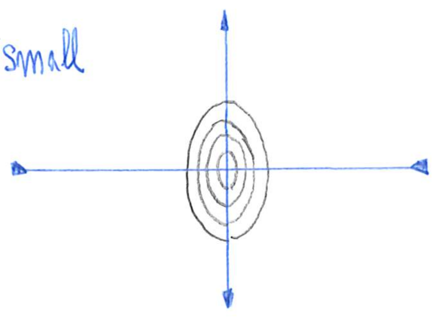
The conclusion of thm 1 is very intuitive:

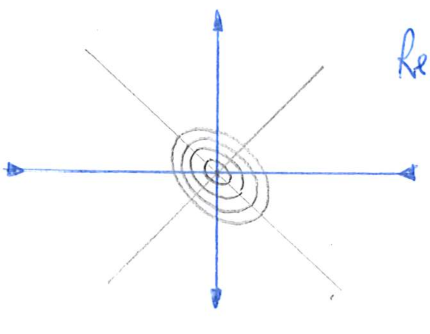
- due to diffusion, the point vortex evolves into a Lamb-Oseen vortex
- due to the external field, the vortex center follows the ODE $z' = f(z,t)$
- both effects do not interact at this level of approximation.

However, it is not clear how to prove thm 1 without computing a much more precise approximation of the solution, taking into account the deformation of the vortex under the strain of the external field f.

<u>Deformation of a vortex in a strain field</u> $f(x) = \gamma(-x_1, x_2)$

Re small





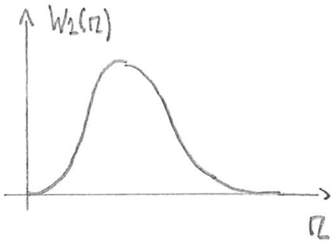
Re large

Ansatz for a Gaussian vortex of circulation $\Gamma > 0$ and core size $\ell > 0$ located at point $z \in \mathbb{R}^2$, in the external field f:

$$\omega_{app}(\Gamma, \ell, z, f; x) = \frac{\Gamma}{\ell^2} \Omega_0\left(\frac{x-z}{\ell}\right) + W_2(r)\left(a(z)\sin(2\vartheta) - b(z)\cos(2\vartheta)\right) \qquad (A)$$

where:

- $\xi = \dfrac{x-z}{\ell} = (r\cos\vartheta, r\sin\vartheta)$ (rescaled variable in the vortex core)
- $a(z) = \frac{1}{2}(\partial_1 f_1 - \partial_2 f_2)(z), \quad b(z) = \frac{1}{2}(\partial_1 f_2 + \partial_2 f_1)(z)$ (strain rates)
- $W_2(r) > 0$ solution of a differential equation



$$\begin{cases} W_2(r) = O(r^2), & r \to 0 \\ W_2(r) \simeq (r^4/8)\, e^{-r^2/4}, & r \to +\infty \end{cases}$$

Rem: $\dfrac{1}{\Gamma}\displaystyle\int_{\mathbb{R}^2} \frac{\Gamma}{\ell^2}\Omega_0\left(\frac{x-z}{\ell}\right) dx = 1$

$$\frac{1}{\Gamma}\int_{\mathbb{R}^2} \left|W_2\left(\frac{|x-z|}{\ell}\right)\right| \left|a\sin(2\vartheta) - b\cos(2\vartheta)\right| dx \le C\,\frac{\ell^2}{\Gamma T_0} = C\,\frac{\ell^2}{d^2}$$

$\Rightarrow$ the correction term in (A) is relatively smaller if $\dfrac{\ell}{d} \ll 1$, $\quad d = \sqrt{\Gamma T_0}$.

Theorem 2: Fix $\Gamma > 0$ and $z_0 \in \mathbb{R}^2$. There exist $K_0 > 0$ and $\delta_0 > 0$ such that, if $0 < \nu < \Gamma\delta_0$, the unique solution of (V_f) with initial data $\omega_0 = \Gamma\delta_{z_0}$ satisfies:

$$\frac{1}{\Gamma}\int_{\mathbb{R}^2} \left|\omega(x,t) - \omega_{app}(\Gamma, \sqrt{\nu t}, z(t), f(t); x)\right| dx \le K_0\, \varepsilon(t)^2(\varepsilon(t) + \delta), \quad 0 < t \le T$$

where $\varepsilon(t) = \sqrt{\nu t}/d$ and $z(t)$ is the solution of the modified ODE:

$$z'(t) = f(z(t), t) + \nu t\, \Delta f(z(t), t), \qquad z(0) = z_0.$$

Rem: Theorem 2 $\Rightarrow$ Theorem 1!

Remarks on Thm 2:

1) The approximation is much more precise than in Thm 1: $O(\varepsilon^3)$ instead of $O(\varepsilon)$. In particular the $O(\varepsilon^2)$ term in ω_{app} (involving w_2 and a, b) cannot be absorbed in the error terms $\Rightarrow$ our result allows us to describe to leading order the deformation of the vortex.

2) The ODE for $z(t)$ contains a (small) additional term $\nu t \Delta f(z(t),t)$, which vanishes if f is irrotational: $\Delta f = \nabla^\perp(\partial_1 f_2 - \partial_2 f_1)$. Otherwise it modifies the solution $z(t)$ by $O(\varepsilon^2 d)$.

3) The origin of the correction term $\nu t \Delta f$ can be understood as follows. To leading order we have:

$$\omega(x,t) = \frac{\Gamma}{\nu t}\,\Omega_0\Big(\frac{x - z(t)}{\sqrt{\nu t}}\Big).$$

Taking this approximation for granted we compute the center of vorticity:

- $$\bar{x}(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\,\omega(x,t)\,dx = \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\,\frac{\Gamma}{\nu t}\,\Omega_0\Big(\frac{x-z(t)}{\sqrt{\nu t}}\Big)\,dx \qquad x = z(t) + \sqrt{\nu t}\,\xi$$

$$= \int_{\mathbb{R}^2}\big(z(t) + \sqrt{\nu t}\,\xi\big)\,\Omega_0(\xi)\,d\xi = z(t)$$

- $$\bar{x}'(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} f(x,t)\,\omega(x,t)\,dx = \int_{\mathbb{R}^2} f(z(t) + \sqrt{\nu t}\,\xi, t)\,\Omega_0(\xi)\,d\xi$$

$$= \int_{\mathbb{R}^2}\Big\{ f(z(t),t) + \sqrt{\nu t}\,Df(z(t),t)[\xi] + \frac{1}{2}(\nu t)\,D^2 f(z(t),t)[\xi,\xi] + O((\nu t)^{3/2})\Big\}\,\Omega_0(\xi)\,d\xi$$

$$= f(z(t),t) + 0 + \nu t\,\Delta f(z(t),t) + O((\nu t)^{3/2}),$$

because

$$\int_{\mathbb{R}^2} \xi_1^2\,\Omega_0(\xi)\,d\xi = \int_{\mathbb{R}^2}\xi_2^2\,\Omega_0(\xi)\,d\xi = 2, \quad \int_{\mathbb{R}^2}\xi_1\xi_2\,\Omega_0(\xi)\,d\xi = 0.$$

4) Under the assumptions of Thm 2, one can show that $|\bar{x}(t) - z(t)| \le C d\,\varepsilon^3(\varepsilon + \delta)$.

III The well-prepared case: proof of Theorem 2

Step 1: Self-similar variables

The solution of (Vf) that we consider is very singular near initial time.
To desingularize the Cauchy problem, we make the change of variables:

$$\left[\; \omega(x,t) = \frac{\Gamma}{\nu t}\,\Omega\Big(\frac{x-z(t)}{\sqrt{\nu t}}, t\Big), \quad u(x,t) = \frac{\Gamma}{\sqrt{\nu t}}\,U\Big(\frac{x-z(t)}{\sqrt{\nu t}}, t\Big),\right.$$

where $z(t)$ is the vortex center (to be determined later). The new space variable

$$\xi = \frac{x-z(t)}{\sqrt{\nu t}}$$

is centered at $z(t)$, and measures distances in units of the diffusion length $\sqrt{\nu t}$.

Since $\int_{\mathbb{R}^2}\omega(x,t)\,dx = \Gamma$, we have $\int_{\mathbb{R}^2}\Omega(\xi,t)\,d\xi = 1 \quad \forall t > 0$.

The evolution equation (Vf) becomes: $\quad U = BS[\Omega]$!

$$\left[\; t\partial_t\Omega(\xi,t) + \frac{1}{\delta}\Big(U(\xi,t) + E(\xi,t)\Big)\cdot\nabla\Omega(\xi,t) = \mathcal{L}\,\Omega(\xi,t) \qquad \begin{matrix}\xi\in\mathbb{R}^2\\ t>0\end{matrix} \qquad (\Omega_{eq})\right.$$

where:

- $\mathcal{L} = \Delta_\xi + \frac{1}{2}\xi\cdot\nabla_\xi + 1$ is the rescaled diffusion operator;
- $\frac{1}{\delta}E(\xi,t) = \sqrt{\frac{t}{\nu}}\Big(\underbrace{f(z(t)+\sqrt{\nu t}\,\xi, t)}_{\text{ext. field}} - \underbrace{z'(t)}_{\text{vortex speed}}\Big)$.

Recalling that $\delta = \frac{\nu}{\Gamma}$, $\varepsilon = \frac{\sqrt{\nu T}}{d}$, $d = \sqrt{\Gamma T_0}$, we see that

$$\delta\sqrt{\frac{t}{\nu}} = \frac{\delta\sqrt{\nu t}}{\nu} = \frac{\sqrt{\nu t}}{\Gamma} = \frac{\sqrt{\nu t}}{d^2}\,T_0 = \underbrace{\frac{T_0}{d}}_{\text{typical speed}}\,\varepsilon$$

hence:

$$E(\xi,t) = \frac{\varepsilon T_0}{d}\Big(f(z(t)+\overbrace{\sqrt{\nu t}}^{=\varepsilon d}\,\xi, t) - z'(t)\Big).$$

The vortex speed $z'(t)$ will be chosen so as to make $E(\xi,t)$ as $\underbrace{\text{small as possible}}_{O(\varepsilon^2)}$.
The initial position is $z(0) = z_0 \in \mathbb{R}^2$.

Remark: The Cauchy pb for (Ωeq) is not well-posed at initial time $t=0$, because one has $t\partial_t\Omega$ instead of $\partial_t\Omega$ in the right-hand side!
In fact, setting formally $t=0$ in (Ωeq) and recalling that $\varepsilon(0)=0$, we get

$$\frac{1}{\delta}\, U\cdot\nabla\Omega = \mathcal{L}\Omega, \quad U = BS[\Omega]$$

This is the eq. satisfied by the profile Ω of a self-similar solution of the vorticity eq. (VE) with $f=0$. If $\Omega \in L^1(\mathbb{R}^2)$, it was shown by ThG & Wayne (2005) that necessarily $\Omega = \alpha\,\Omega_0$ (the Gaussian vortex), $\alpha\in\mathbb{R}$.
Normalization forces $\alpha = 1$, hence the only possible initial data are:

$$\Omega_0(\xi) = \frac{1}{4\pi}\, e^{-|\xi|^2/4}, \qquad U_0(\xi) = \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\left(1-e^{-|\xi|^2/4}\right).$$

Step 2: Construction of an approximate solution

We look for an approximate solution of (Ωeq) in the form:

$$\begin{cases} \Omega_{app}(\xi,t) = \Omega_0(\xi) + \varepsilon(t)^2\,\Omega_2(\xi,t) + \varepsilon(t)^3\,\Omega_3(\xi,t) + \varepsilon(t)^4\,\Omega_4(\xi,t) \\ U_{app}(\xi,t) = U_0(\xi) + \varepsilon(t)^2\,U_2(\xi,t) + \varepsilon(t)^3\,U_3(\xi,t) + \varepsilon(t)^4\,U_4(\xi,t) \end{cases} \qquad (1)$$

where $U_i = BS[\Omega_i]$ and the profiles $\Omega_2, \Omega_3, \Omega_4$ have to be determined.

Remarks:
- expansion in $\varepsilon(t) = \frac{\sqrt{\nu t}}{d}$, but starts at order ε^2. Order 4 is sufficient!
- all quantities also depend on $\delta = \frac{\nu}{\Gamma}$ (not indicated).

We also have to expand the quantity $E(\xi,t)$ in (Ωeq). Anticipating that $z'(t) = f(z(t),t) + \nu t\,\Delta f(z(t),t)$, we obtain by Taylor expansion:

$$E(\xi,t) = \varepsilon^2 E_2(\xi,t) + \varepsilon^3 E_3(\xi,t) + \varepsilon^4 E_4(\xi,t) + O(\varepsilon^5), \text{ where}$$

$$\begin{cases} E_2(\xi,t) = T_0\, Df(z(t),t)[\xi] \\ E_3(\xi,t) = T_0 d\left(\frac{1}{2} D^2 f(z(t),t)[\xi,\xi] - \Delta f(z(t),t)\right) \qquad (2) \\ E_4(\xi,t) = \frac{1}{6} T_0 d^2 D^3 f(z(t),t)[\xi,\xi,\xi]. \end{cases}$$

We now replace (1) and (2) into (Ωeq) and determine the profiles Ω_i order by order.

Calculations at order 2: N.B. $t\partial_t \varepsilon = \varepsilon/2$, $t\partial_t \varepsilon^2 = \varepsilon \ldots$

- $t\partial_t \Omega_{app} = \varepsilon^2 (t\partial_t \Omega_2 + \Omega_2) + O(\varepsilon^3)$
- $\mathcal{L}\Omega_{app} = \varepsilon^2 \mathcal{L}\Omega_2 + O(\varepsilon^3)$
- $U_{app}\cdot\nabla\Omega_{app} = \varepsilon^2 (U_0\cdot\nabla\Omega_2 + U_2\cdot\nabla\Omega_0) + O(\varepsilon^3)$
- $E\cdot\nabla\Omega_{app} = \varepsilon^2 E_2\cdot\nabla\Omega_0 + O(\varepsilon^3)$

Summarizing, we get at order ε^2: N.B. $\delta t = \varepsilon^2 T_0 = O(\varepsilon^2)$!

$$\delta(1-\mathcal{L})\Omega_2 + \Lambda\Omega_2 + E_2\cdot\nabla\Omega_0 = 0 \qquad (\Omega_2)$$

where Λ is the integro-differential operator defined by:

$$\Lambda\Omega = U_0\cdot\nabla\Omega + BS[\Omega]\cdot\nabla\Omega_0.$$

We shall see that (Ω_2) determines univoquely the profile Ω_2. Similar calculations allow one to compute Ω_3 and Ω_4 too.

Step 3: Inverting the linear operator Λ

It is convenient to introduce the Hilbert space Y defined by

$$Y = \left\{ \Omega\in L^2(\mathbb{R}^2)\,;\ \int_{\mathbb{R}^2} \Omega(\xi)^2 e^{|\xi|^2/4}\, d\xi < \infty \right\}.$$

We equip Y with the natural scalar product.

We recall the remarkable properties of the operators $\mathcal{L}$ and Λ acting on Y.

Prop 1: $\mathcal{L}$ is self-adjoint in Y with discrete spectrum:

$$\sigma(\mathcal{L}) = \{0, -1/2, -1, -3/2, \dots\}.$$ Moreover:

- $\mathrm{Ker}(\mathcal{L})$ is spanned by Ω_0;
- $\mathrm{Ker}(\mathcal{L}+1/2)$ is spanned by $\partial_1\Omega_0$ and $\partial_2\Omega_0$;
- $\mathrm{Ker}(\mathcal{L}+m/2)$ is spanned by $\partial^\alpha\Omega_0$ with $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$, $\alpha_1+\alpha_2 = m$.

Proof: $$e^{|\xi|^2/8}\,\mathcal{L}\,e^{-|\xi|^2/8} = \Delta - \frac{|\xi|^2}{16} + \frac{1}{2}$$; harmonic oscillator in $\mathbb{R}^2$! □

Prop 2: Λ is skew-adjoint in Y, and

$$\mathrm{Ker}(\Lambda) = Y_0 \oplus \{\beta_1\partial_1\Omega_0 + \beta_2\partial_2\Omega_0 \,;\, \beta_1,\beta_2 \in \mathbb{R}\}.$$

Proof: Cf. Gallay-Wayne 2005, Maekawa 2011. (Y_0 = radially symmetric elements of Y)

Returning to (Ω_2): The operator $\delta(1-\mathcal{L}) + \Lambda$ is invertible in Y $\forall \delta > 0$, but $\|(\delta(1-\mathcal{L})+\Lambda)^{-1}\|_{Y\to Y} \le C\delta^{-1}$ in general. To have a solution that depends regularly of δ, we need to check that $E_2\cdot\nabla\Omega_0 \in \mathrm{Ran}(\Lambda)$.

By a direct calculation:

$$E_2\cdot\nabla\Omega_0 = -T_0\,\frac{|\xi|^2}{2}\,\Omega_0\,\big(a(t)\cos(2\vartheta) + b(t)\sin(2\vartheta)\big),$$ where

$$a(t) = \frac{1}{2}(\partial_1 f_1 - \partial_2 f_2)(z(t),t), \quad b(t) = \frac{1}{2}(\partial_1 f_2 + \partial_2 f_1)(z(t),t).$$

Thus $E_2\cdot\nabla\Omega_0 \in \mathrm{Ker}(\Lambda)^\perp \underset{\uparrow}{=} \mathrm{Ran}(\Lambda)$, and in fact $E_2\cdot\nabla\Omega_0 \in \mathrm{Ran}(\Lambda)$.

Λ is skew-adjoint

- $\exists!\ \bar\Omega_2 \in \mathrm{Ker}(\Lambda)^\perp$ s.th. $\Lambda\bar\Omega_2 + E_2\cdot\nabla\Omega_0 = 0$
- $\exists!\ \tilde\Omega_2 \in \mathrm{Ker}(\Lambda)^\perp$ s.th. $\Lambda\tilde\Omega_2 + (1-\mathcal{L})\bar\Omega_2 = 0$

Setting $\underline{\Omega_2 = \bar\Omega_2 + \delta\tilde\Omega_2}$, we conclude $\underline{\delta(1-\mathcal{L})\Omega_2 + \Lambda\Omega_2 + E_2\cdot\nabla\Omega_0} = \underbrace{\delta^2(1-\mathcal{L})\tilde\Omega_2}_{\ll 1}$

Step 4: Estimate on the remainder

Having constructed the profiles $\Omega_2, \Omega_3, \Omega_4$ we consider the remainder

$$R_{app} := \delta\left(t\partial_t\Omega_{app} - \mathcal{L}\Omega_{app}\right) + \left(U_{app} + E\right)\cdot\nabla\Omega_{app}.$$

<u>Lemma:</u> $\exists N \in \mathbb{N}$ $\exists C > 0$ such that

$$|R_{app}(\xi,t)| \leq C\left(\varepsilon^5 + \delta^2\varepsilon^2\right)(1+|\xi|)^N e^{-|\xi|^2/4}.$$

$\uparrow$ because we used a 4^{th} order expansion

<u>Rem:</u> To construct the profile Ω_3, one needs that $E_3\cdot\nabla\Omega_0 \in \mathrm{Ran}(\Lambda)$.
This is not the case if $z'(t) = f(z(t),t)$ $\Longrightarrow$ we added the small correction $\nu t\Delta f$ in the evolution eq. for $z(t)$.

$\Longrightarrow$ The vortex speed $z'(t)$ is chosen so as to ensure <u>solvability conditions</u> in the elliptic eq. for the vortex profiles $\Omega_3, \Omega_5, \Omega_7 \ldots$

<u>Rem:</u> To leading order we have

$$\Omega_{app}(\xi,t) = \Omega_0(\xi) + \varepsilon^2\bar{\Omega}_2(\xi,t) + O(\varepsilon^3 + \varepsilon^2\delta) \quad \Longrightarrow \text{expression of } \omega_{app}!$$

A direct calculation shows that

$$\bar{\Omega}_2(\xi,t) = T_0 W_2(|\xi|)\left(a(t)\sin(2\vartheta) - b(t)\cos(2\vartheta)\right)$$

where:

- $W_2(r) = h(r)\left(\varphi_2(r) + \frac{r^2}{2}\right), \quad h(r) = \frac{r^2/4}{e^{r^2/4} - 1}$ (Arnold function)
- $\varphi_2(r)$ is the unique sol. of the ODE:

$$-\varphi_2'' - \frac{1}{r}\varphi_2' + \left(\frac{4}{r^2} - h\right)\varphi_2 = \frac{r^2}{2}h(r)$$

such that $\varphi_2(r) = O(r^2)$ as $r \to 0$ and $\varphi_2(r) = O(1/r^2)$ as $r \to +\infty$.

Step 5 : Correction to the approximate solution

Finally we decompose the exact solution of (Ω_{eq}) as:

$$\Omega(\xi,t) = \Omega_{app}(\xi,t) + \delta\, w(\xi,t), \quad U(\xi,t) = U_{app}(\xi,t) + \delta\, v(\xi,t)$$

The factor δ here is a choice, anticipating that the correction should be small.

The equation for w takes the form: $\quad v = BS[w]$

$$t\partial_t w + \frac{1}{\delta}\Lambda w + \frac{1}{\delta}A[w] + B[w,w] = \mathcal{L}w - \frac{1}{\delta^2}R_{app} \qquad (W_{eq})$$

where:

- $\mathcal{L} = \Delta + \frac{1}{2}\xi\cdot\nabla + 1$: diffusion operator
- $\Lambda w = U_0\cdot\nabla w + BS[w]\cdot\nabla\Omega_0$: linearization of Euler at Ω_0
- $A[w] = (U_{app} - U_0)\cdot\nabla w + BS[w]\cdot\nabla(\Omega_{app} - \Omega_0) + E(f,z)\cdot\nabla w$
- $B[w,w] = BS[w]\cdot\nabla w$ $\qquad$ (- - - - : added and subtracted terms)

and R_{app} is the remainder defined in step 4.

Rem:

- there is no factor $\frac{1}{\delta}$ in front of $B[w,w]$ because the perturbation is δw
- for the same reason, there is a factor $\frac{1}{\delta^2}$ in front of R_{app}.
- Eq. (W_{eq}) has to be solved with zero initial data, because $\Omega(\xi,0) = \Omega_{app}(\xi,0) = \Omega_0(\xi)$. Also $\int_{\mathbb{R}^2} w\, d\xi = 0$.
- The source term is small:

$$\left|\frac{1}{\delta^2}R_{app}\right| \leqslant C\left(\frac{\varepsilon^5}{\delta^2} + \varepsilon^2\right)(1+|\xi|)^N e^{-|\xi|^2/4}, \quad \text{and}$$

$$\varepsilon^2 = \delta\frac{t}{T_0} \implies \frac{\varepsilon^5}{\delta^2} = \varepsilon\left(\frac{t}{T_0}\right)^2 \ll 1.$$

The main difficulty is to prove that the vorticity correction $w(\xi,t)$ does not get amplified by the linear terms in (W eq) involving $1/\delta$. This can be done using energy estimates.

Simple case: $T \ll T_0 \implies$ energy estimates in Y

$$\begin{cases} \mathcal{E}[w] = \|w\|_Y^2 = \int_{\mathbb{R}^2} |w|^2 e^{|\xi|^2/4}\, d\xi \\ \mathcal{F}[w] = \int_{\mathbb{R}^2} \left\{ |\nabla w|^2 + |\xi|^2 |w|^2 + |w|^2 \right\} e^{|\xi|^2/4}\, d\xi \geq \mathcal{E}[w] \end{cases}$$

$$\frac{1}{2} t\, \partial_t \mathcal{E}[w] = \langle w, t\partial_t w \rangle_Y$$

$$= \langle w, \mathcal{L}w \rangle_Y - \frac{1}{\delta} \langle w, A[w] \rangle_Y - \langle w, B[w,w] \rangle_Y - \frac{1}{\delta^2} \langle w, R_{app} w \rangle_Y .$$

N.B. $\frac{1}{\delta} \langle w, \Lambda w \rangle_Y = 0$ because Λ is skew-symmetric in Y

- diffusion term: since $\int w\, d\xi = 0$, $\exists\, k > 0$ s.that $\langle w, \mathcal{L}w \rangle_Y \leq -k\mathcal{F}[w]$.
- advection terms: $p := e^{|\xi|^2/4}$, $\nabla p = (\xi/2)p$

$$\int_{\mathbb{R}^2} p w (U_{app} - U_0) \cdot \nabla w \, d\xi = -\frac{1}{2} \int_{\mathbb{R}^2} w^2 (U_{app} - U_0) \cdot \nabla p \, d\xi$$

$$= \frac{1}{4} \int_{\mathbb{R}^2} w^2 \underbrace{(U_{app} - U_0) \cdot \xi}_{\leq C\varepsilon^2}\, p\, d\xi$$

$$\implies \frac{1}{\delta} \left| \langle w, (U_{app} - U_0) \cdot \nabla w \rangle_Y \right| \leq C \frac{\varepsilon^2}{\delta} \|w\|_Y^2 = C \frac{t}{T_0} \|w\|_Y^2$$

Idem: $\frac{1}{\delta} \left| \langle w, v \cdot \nabla (\Omega_{app} - \Omega_0) \rangle_Y \right| \leq C \frac{\varepsilon^2}{\delta} \|w\|_Y^2$

$$|E(f, z; \xi, t)| \leq \varepsilon^2 |\xi| + C\varepsilon^3$$

$$\implies \frac{1}{\delta} \langle w, E(f,z) \cdot \nabla w \rangle \leq \frac{\varepsilon^2}{\delta} \mathcal{F}[w] = \frac{t}{T_0} \mathcal{F}[w].$$

- nonlinear term: classical estimates

$$|\langle W, V\cdot\nabla W\rangle_Y| \leq C\,\mathcal{F}[W]^{1/2}\,\mathcal{E}[W]$$

- remainder term:

$$\frac{1}{\delta^2}|\langle W, R_{app}\rangle| \leq C\left(\frac{\varepsilon^5}{\delta^2}+\varepsilon^2\right).$$

Summarising: $\exists\, k>0\ \exists\, C>0$

$$t\partial_t\,\mathcal{E}[W] + 2\left(k-\frac{t}{T_0}\right)\mathcal{F}[W] \leq C\frac{t}{T_0}\mathcal{E}[W] + C\mathcal{F}[W]^{1/2}\mathcal{E}[W] + C\left(\frac{\varepsilon^5}{\delta^2}+\varepsilon^2\right)^2.$$

Assuming that $t/T_0 \leq T/T_0 \leq k/2$, we can integrate the differential inequality to obtain the estimate:

$$\|W(\tau)\|_Y = \mathcal{E}[W(\cdot,\tau)]^{1/2} \leq C\left(\frac{\varepsilon^5}{\delta^2}+\varepsilon^2\right), \quad 0<t\leq T.$$

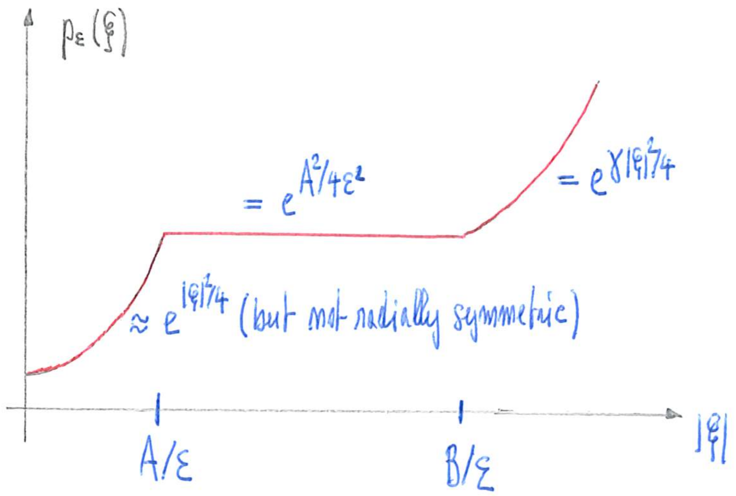
In particular $\delta\|W(\tau)\|_Y \leq C\left(\frac{\varepsilon^5}{\delta}+\delta\varepsilon^2\right) \leq C\varepsilon^2(\varepsilon+\delta)$. Since $\|W(\cdot,\tau)\|_{L^1} \leq C\|W(\cdot,\tau)\|_Y$, this gives the estimate in Thm 2.

General case: T/T_0 potentially large

- The advection term $\frac{1}{\delta}E(\xi,\tau)\cdot\nabla W$ cannot be controled as before
  $\Longrightarrow$ One has to modify the weight function so that it is constant on the streamlines of $U_0 + E(\xi,\tau)$. This can be done if $|\xi| \leq A/\varepsilon$, $A \ll 1$
- For $A \leq |\xi|\varepsilon \leq B$ with $B \gg 1$, the weight function is taken constant.
- For $|\xi| \geq B/\varepsilon$ one takes the weight $\exp(\gamma|\xi|^2/4)$ with $\gamma = A^2/B^2 \ll 1$.

The calculations are (much) more complicated, but lead to a similar differential inequality for $\mathcal{E}[W]$.

The weight function in the general case



$A \ll 1 \ll B$

$\gamma = A^2/B^2 \ll 1$

$$\mathcal{E}_\varepsilon[W] = \int_{\mathbb{R}^2} |W(\xi)|^2 p_\varepsilon(\xi)\, d\xi\,, \quad \mathcal{F}_\varepsilon[W] = \int_{\mathbb{R}^2} \left\{ |\nabla W|^2 + \chi_\varepsilon W^2 + W^2 \right\} p_\varepsilon(\xi)\, d\xi$$

Inner region: $|\xi| \lesssim A/\varepsilon \qquad p_\varepsilon(\xi,t) = \exp\left(q_\varepsilon(\xi,t)\right), \quad \chi_\varepsilon = |\xi|^2$

$$\left[\quad q_\varepsilon(\xi,t) = \frac{|\xi|^2}{4} + \frac{\varepsilon^2 T_0}{4 V_0(\xi)} \left(b(t)(\xi_1^2 - \xi_2^2) - 2a(t)\xi_1\xi_2 \right), \quad V_0(\xi) = \frac{1}{2\pi|\xi|^2}\left(1 - e^{-|\xi|^2/4}\right). \right.$$

Has the property that $U_0 \cdot \nabla q_\varepsilon + \varepsilon^2 E_2 \cdot \nabla q_\varepsilon = O\left((|\xi|^2 + |\xi|^4)\varepsilon^4\right)$; partial cancellation.

or: $U_0 \cdot \nabla q_\varepsilon + \varepsilon^2 E_2 \cdot \nabla q_0 = 0.$

Intermediate region: $p_\varepsilon(\xi) = \exp(A^2/4\varepsilon^2), \qquad A/\varepsilon \lesssim |\xi| \leq B/\varepsilon, \quad \chi_\varepsilon = A^2/\varepsilon^2$

$\Longrightarrow$ all advection terms give zero contribution to $t\partial_t \mathcal{E}_\varepsilon[W]$.

Far-field region: $|\xi| \gtrsim B/\varepsilon, \quad p_\varepsilon(\xi) = e^{\gamma|\xi|^2/4}, \quad \chi_\varepsilon = \gamma|\xi|^2.$

This region is dominated by the diffusion operator $\mathcal{L}$.

Since $\frac{1}{\delta}|E(\xi,t)| \leq C\frac{\varepsilon}{\delta} = \frac{C}{\varepsilon}\frac{t}{T_0} \leq \frac{C}{\varepsilon}\frac{T}{T_0}$,

the contribution of $\frac{1}{\delta}E(\xi,t)\cdot\nabla$ is smaller than that of $\frac{1}{2}\xi\cdot\nabla$,

provided $B \gg 1$. $\square$

IV The ill-prepared case : Gaussian initial data

We now investigate, at least on an example, what happens if we consider sharply concentrated initial data which differ from a point vortex. To simplify the comparison with Thm 2, we choose some time $t_0 \in (0,T)$ and we assume that:

$$\omega(x,t_0) = \frac{\Gamma}{\nu t_0}\,\Omega_0\!\left(\frac{x-z_0}{\sqrt{\nu t_0}}\right), \quad x \in \mathbb{R} \qquad (*)$$

for some $z_0 \in \mathbb{R}^2$.

Thm 3: Fix $\Gamma > 0$, $z_0 \in \mathbb{R}^2$, $0 < t_0 < T$. There exist positive constants K_0, δ_0, c_0 such that, if $0 < \nu/\Gamma < \delta_0$, the solution of (Vf) with initial data $(*)$ at time t_0 satisfies, for $t_0 \leq t \leq T$:

$$\frac{1}{\Gamma}\int_{\mathbb{R}^2}\left|\omega(x,t) - \omega_{app}(\Gamma, \sqrt{\nu t}, z(t), f(\cdot,t); x)\right| dx \leq K_0\,\varepsilon^2\left(\delta^{1/6}\log\frac{1}{\delta} + \left(\frac{t_0}{t}\right)^{\beta}\right)$$

where $\delta = \nu/\Gamma$, $\varepsilon = \sqrt{\nu t}/d$, $\beta = c_0\,\delta^{-1/3}$, and z is the unique solution of the ODE

$$z'(t) = f(z(t),t) + \nu t\,\Delta f(z(t),t), \qquad z(t_0) = z_0.$$

Remark: At time t_0, in view of $(*)$, we have

$$\frac{1}{\Gamma}\left\|\omega(\cdot,t_0) - \omega_{app}(\Gamma,\sqrt{\nu t_0}, z_0, f(\cdot,t_0);\cdot)\right\|_{L^1} = O(\varepsilon(t_0)^2) \quad \text{if } a(t_0), b(t_0) \neq (0,0).$$

If $t \geq t_0 + O(\delta^{1/3}\log\frac{1}{\delta})$, the estimate in thm 3 shows that

$$\frac{1}{\Gamma}\left\|\omega(\cdot,t_0) - \omega_{app}(\Gamma,\sqrt{\nu t}, z(t), f(\cdot,t))\right\|_{L^1} \leq K\,\varepsilon^2\,\delta^{1/6}\log\frac{1}{\delta} \ll K\,\varepsilon^2.$$

During that short interval, the initially radially symmetric vortex adapts its shape to the strain of the external field. The relaxation rate $\beta = c_0\,\delta^{-1/3}$ becomes very large as $\delta \to 0+$: enhanced dissipation effect!

The strategy of the proof of Thm 3 is very similar to that of Thm 2. We use self-similar variables and decompose the vorticity as above:

$$\Omega(\xi,t) = \Omega_{app}(\xi,t) + \delta\, w(\xi,t), \quad t \geq t_0.$$

We thus arrive at the evolution equation

$$t\partial_t w + \frac{1}{\delta}\Lambda w + \frac{1}{\delta}A[w] + B[w,w] = \mathcal{L}w - \frac{1}{\delta^2}R_{app}, \qquad t \geq t_0. \qquad (\text{Weq})$$

⚠ The Cauchy pb for (Weq) is well-posed at time $t_0 > 0$! No problem with the time derivative $t\partial_t$ $\Rightarrow$ we can impose arbitrary initial data.

In view of $(*)$ we have:

$$w(\xi,t_0) = \varphi_0(\xi) := \frac{1}{\delta}\Big(\Omega_0(\xi) - \Omega_{app}(\xi,t_0)\Big) = O\Big(\frac{\varepsilon_0^2}{\delta}\Big) = O\Big(\frac{t_0}{T_0}\Big)$$

where $\varepsilon_0 = \sqrt{\nu t_0}/d \ll 1$. This is in contrast with the proof of Thm 2 where $w(\xi,0) = 0$.

The idea is now to make another decomposition:

$$w(\xi,t) = w_0(\xi,t) + \tilde{w}(\xi,t), \qquad v(\xi,t) = v_0(\xi,t) + \tilde{v}(\xi,t),$$

where:

i) w_0 solves a linear equation with nonzero initial data:

$$\begin{cases} t\partial_t w_0(\xi,t) + \frac{1}{\delta}\Lambda w_0(\xi,t) = \mathcal{L}w_0(\xi,t), & t \geq t_0 \\ w_0(\xi,t_0) = \varphi_0(\xi) \end{cases}$$

ii) $\tilde{w}$ solves a nonlinear equation with zero initial data:

$$\begin{cases} t\partial_t \tilde{w} + \frac{1}{\delta}\Lambda\tilde{w} + \frac{1}{\delta}A[w_0+\tilde{w}] + B[w_0+\tilde{w}, w_0+\tilde{w}] = \mathcal{L}\tilde{w} - \frac{1}{\delta^2}R_{app} \\ \tilde{w}(\xi,t_0) = 0. \end{cases}$$

Idea: w_0 decays rapidly to zero, $\tilde{w}$ stays small.

Step 1: Enhanced dissipation estimates

To study the evolution equation for W_0, it is convenient to introduce the logarithmic time

$$\tau = \log(t/t_0) \iff t = t_0 e^{\tau} \qquad \begin{cases} t \geq t_0 \\ \tau \geq 0 \end{cases}$$

The equation becomes:

$$\partial_\tau W_0(\xi,\tau) + \frac{1}{\delta}\Lambda W_0(\xi,\tau) = \mathcal{L} W_0(\xi,\tau) \qquad \xi \in \mathbb{R}^2,\ \tau \geq 0.$$

The solution can be written in the form:

$$W_0(\tau) = e^{\tau(\mathcal{L} - 1/\delta \Lambda)} \varphi_0, \quad \tau \geq 0.$$

Remark: Since $W_0 \in Y$ and $\int_{\mathbb{R}^2} W_0 \, d\xi = 0$, we have

$$\frac{1}{2}\frac{d}{d\tau}\|W_0(\tau)\|_Y^2 = \langle W_0(\tau), \partial_\tau W_0(\tau)\rangle_Y$$

$$= \langle W_0(\tau), \mathcal{L} W_0(\tau)\rangle_Y - \frac{1}{\delta}\underbrace{\langle W_0(\tau), \Lambda W_0(\tau)\rangle_Y}_{=0}$$

Λ skew-symmetric!

$$\leq -\frac{1}{2}\|W_0(\tau)\|_Y^2, \quad \text{cf. Prop 1. on page 17,}$$

$$\Longrightarrow \|W_0(\tau)\|_Y \leq e^{-\tau/2}\|\varphi_0\|_Y, \quad \forall \tau \geq 0.$$

This simple energy estimate does not take into account the "mixing effect" of the skew-symmetric operator Λ, which is important if $\delta \ll 1$.

Thm: (Te Li, Dongyi Wei, Zhifei Zhang, 2020)

There exist positive constants $C_0 > 0$, $c_0 > 0$ such that; for $0 < \delta < 1$:

$$\|W_0(\tau)\|_Y \leq C_0 e^{-c_0 \delta^{-1/3}\tau}\|\varphi_0\|_Y, \quad \forall \tau \geq 0$$

⚠ $\varphi_0 \in \mathrm{Ker}(\Lambda)^\perp$!

The decay rate here is $c_0\delta^{-1/3} \gg 1$ if δ is small: <u>enhanced dissipation effect</u>.

N.B. An exponential decay in the logarithmic time τ corresponds to a polynomial decay in the original time t.

Remark: The decay rate $c_0\delta^{-1/3}$ is optimal for general initial data, but might be improved for the particular data φ_0 considered here. To compare with:

- decay rate for monotone shear flows: $\approx \delta^{-2/3}$ (Couette)
- decay rate for shear flows with a non-degenerate stagnation point: $\approx \delta^{-1/2}$.

Enhanced dissipation estimates can be used to study the size of the basin of attraction of the Lamb-Oseen vortex, see ThG 2018.

Corollary 1 (easy): $\displaystyle\int_0^\infty \|w_0(\tau)\|_Y^2\,d\tau \leq C\,\delta^{1/3}$.

N.B. No such estimate exists for $\nabla w_0(\tau)$: One can show that

$$\int_0^\infty \|\nabla w_0(\tau)\|_Y^2\,d\tau \geq \frac{1}{2}\|\varphi_0\|_Y^2 \;!$$

Corollary 2 (more difficult): If $\gamma > 1/8$, there exists $C>0$ such that $\forall \varphi_0 \in \mathrm{Ker}(\Lambda)^\perp$:

$$\int_0^\infty \|\,|\xi|\,w_0(\tau)\|_Y^2\,d\tau \leq C\,\delta^{1/3}\log\left(\frac{1}{\delta}\right)\underbrace{\sup_{\xi\in\mathbb{R}^2} e^{2\gamma|\xi|^2}|\varphi_0(\xi)|^2}_{<\infty \text{ if } \gamma<1/4}.$$

Idea of the proof:

- In the region where $|\xi| \leq N\log(1/\delta)^{1/2}$: use corollary 1
- In the region where $|\xi| > N\log(1/\delta)^{1/2}$: the operator Λ can be replaced by $\Lambda_0 = U_0\cdot\nabla$, because $|\Omega_0| \leq C\,\delta^{N^2/4}$ ($\Rightarrow$ the nonlocal term is small).

The advection operator Λ_0 can be handled by energy estimates for any radial weight.

Step 2 : Energy estimates

We return to the equation for $\tilde{w}$:

$$\begin{cases} t\partial_t \tilde{w} + \frac{1}{\delta}\Lambda\tilde{w} + \frac{1}{\delta}A[w_0+\tilde{w}] + B[w_0+\tilde{w}, w_0+\tilde{w}] = \mathcal{L}\tilde{w} - \frac{1}{\delta^2}R_{app} \\ \tilde{w}(t_0) = 0 \end{cases}$$

The particular case where $w_0 = 0$ has been treated in Thm 2.
All terms involving w_0 are considered as additional source terms, and estimated using Corollaries 1 and 2 above. This leads to an estimate of the form:

$$\mathcal{E}[\tilde{w}(\cdot,t)]^{1/2} \leq C\left(\frac{\varepsilon^5}{\delta^2} + \varepsilon^2\right) + C\delta^{1/6}\left(\log\frac{1}{\delta}\right)^{1/2}\frac{t}{T_0}, \qquad t_0 \leq t \leq T.$$

As in thm 2, we can take $\mathcal{E}[\tilde{w}] = \|\tilde{w}\|_Y^2$ if $T \ll T_0$, but in the general case a more complicated energy functional has to be used.

Summarizing, we have shown:

$$\begin{aligned} \|\Omega(t) - \Omega_{app}(t)\|_{L^1} &= \delta\|w(t)\|_{L^1} \leq \delta\|w_0(t)\|_{L^1} + \delta\|\tilde{w}(t)\|_{L^1} \\ &\leq C\delta\|w_0(t)\|_Y + C\delta\,\mathcal{E}[\tilde{w}(\cdot,t)]^{1/2} \\ &\leq C\delta\left(\frac{t_0}{t}\right)^\beta\frac{t_0}{T_0} + C\delta\frac{t}{T_0}\delta^{1/6}\left(\log\frac{1}{\delta}\right)^{1/2} \\ &\leq C\varepsilon^2\left(\left(\frac{t_0}{t}\right)^\beta + \delta^{1/6}\left(\log\frac{1}{\delta}\right)^{1/2}\right), \qquad \text{where } \beta = c_0\delta^{-1/3} \gg 1. \end{aligned}$$

Returning to the original variables, this implies the approximation property stated in Thm 3. $\square$